INSTITUTE OF MATHEMATICS
of the
Polish Academy of Sciences

ul. Śniadeckich 8, P.O.B. 21, 00-956 Warszawa 10, Poland http://www.impan.gov.pl

IM PAN Preprint 683 (2007)

Stefan Rolewicz

How to define "convex functions" on differentiable manifolds

Published as manuscript

Received 27 June 2007

How to define "convex functions" on differentiable manifolds.

S.Rolewicz
Institute of Mathematics of the Polish Academy of Sciences
Śniadeckich 8, 00-956 Warszawa, P.O.Box 21, POLAND
(e-mail: rolewicz@impan.gov.pl)

Abstract. In the paper a class of families $\mathcal{F}(M)$ of functions defined on differentiable manifolds M with the following properties:

$1_{\mathcal{F}}$. if M is a linear manifold, then $\mathcal{F}(M)$ contains convex functions,

$2_{\mathcal{F}}$. $\mathcal{F}(\cdot)$ is invariant under diffeomorphisms,

$3_{\mathcal{F}}$. each $f \in \mathcal{F}(M)$ is differentiable on a residual set,

is investigated.

Let $(X, \|.\|)$ be a Banach space over reals. Let $f(x)$ be a real-valued convex continuous function defined on an open convex subset $\Omega \subset X$, i.e.

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in \Omega$ and $t,\ 0 \leq t \leq 1$.

We recall that a set $B \subset \Omega$ of second Baire category is called *residual* if its complement $\Omega \setminus B$ is of the first Baire category (i.e. it is a countable union of nowhere dense sets). Mazur (1933) proved that in the case of separable X there is a residual subset A_G such that on the set A_G the function f is Gateaux differentiable. Asplund (1968) showed that if in the dual space X^* there exists an equivalent locally uniformly rotound norm, then there is a residual subset A_F such that on the set A_F the function f is Fréchet differentiable. The spaces X such that for the dual space X^* there exists an equivalent locally uniformly rotound norm are now called *Asplund spaces*. It can be shown that each reflexive space and spaces having separable duals are Asplund spaces. Even more a space X is an Asplund space if and only if each its separable subspace $X_0 \subset X$ has a separable dual (Phelps (1989)).

The aim of this note is to obtain similar results for functions defined on differentiable manifolds. The first problem is how to define "convex function" in this case. For this purpose we shall introduce a class of families $\mathcal{F}(M)$ of functions defined on differentiable manifold M over a Banach space $\mathbf{E}$ with the following properties:

$1_{\mathcal{F}}$. if M is a linear manifold, then $\mathcal{F}(M)$ contains convex functions,

$2_{\mathcal{F}}$. $\mathcal{F}(\cdot)$ is invariant under diffeomorphisms,

$3_{\mathcal{F}}$. each $f \in \mathcal{F}(M)$ is

(a). Fréchet differentiable on a dense G_δ-set provided $\mathbf{E}$ is an Asplund space,

(b). Gateaux differentiable on dense G_δ-set provided $\mathbf{E}$ is separable.

At the beginning we recall the notion of differentiable manifolds.

Let $\mathbf{E}$, $\mathbf{F}$, be Banach spaces over reals. We say that a function $\psi : \mathbf{E} \to \mathbf{F}$ is of the class $C^{1,u}_{\mathbf{E},\mathbf{F}}$ if it is continuously differentiable and moreover that differential $\partial\psi\Big|_x$ is locally uniformly continuous as a function of x in the norm topology. Of course, if $\psi \in C^{1,u}_{\mathbf{E},\mathbf{F}}$, then ψ belongs to the class of continuously differentiable functions, $\psi \in C^{1}_{\mathbf{E},\mathbf{F}}$. The converse is true if $\mathbf{E}$ is finite dimensional.

If $\mathbf{E} = \mathbf{F}$ we denote briefly $C^{1,u}_{\mathbf{E},\mathbf{E}} = C^{1,u}_{\mathbf{E}}$.

Now we shall determine $C^{1,u}_{\mathbf{E}}$-manifold in the classical way (compare Lang (1962)).

Let M be a set. An $C^{1,u}_{\mathbf{E}}$-*atlas* is a collections of pairs (U_i, ϕ_i) (i ranging in some indexing set I) satisfying the following conditions:

AT 1. Each U_i is a subset of M and $\{U_i\}$ covers M, $M \subset \cup_{i\in I} U_i$,

AT 2. Each ϕ_i is a bijection of U_i onto an open subset $\phi_i(U_i)$ of the space $\mathbf{E}$, and for all i, j, $\phi_i(U_i \cap U_j)$ is an open subset of the space $\mathbf{E}$,

AT 3. The map $\phi_j \phi_i^{-1}$ mapping $\phi_i(U_i \cap U_j)$ onto $\phi_j(U_i \cap U_j)$ is of the class $C^{1,u}_{\mathbf{E}}$ for all i, j.

Each pair (U_i, ϕ_i) is called a *chart*. If $x \in U_i$, then the pair (U_i, ϕ_i) is called a *chart at* x.

Observe that AT 3 implies that $\Big(\phi_j \phi_i^{-1}\Big)^{-1} = \phi_i \phi_j^{-1} \in C^{1,u}_{\mathbf{E}}$.

Suppose now that M is a topological space and let U be an open set in M. Suppose that there is a topological isomorphism ϕ mapping U onto an open set $U' \in \mathbf{E}$. We say that (U, ϕ) is *compatible* with the $C^{1,u}_{\mathbf{E}}$-atlas (U_i, ϕ_i) if for all i the maps $\phi_i \phi^{-1}$ and $\phi \phi_i^{-1}$ belong to $C^{1,u}_{\mathbf{E}}$. We say that two $C^{1,u}_{\mathbf{E}}$-atlases are *compatible* if each chart of one is compatible with the other $C^{1,u}_{\mathbf{E}}$-atlas.

A topological space M equipped with $C^{1,u}_{\mathbf{E}}$-atlas (U_i, ϕ_i) we shall call $C^{1,u}_{\mathbf{E}}$-*manifold*.

Let M be a $C^{1,u}_{\mathbf{E}}$-manifold. Let (U_i, ϕ_i) be a $C^{1,u}_{\mathbf{E}}$-atlas on X. Let $f(\cdot)$ be a real-valued function $f(\cdot)$ defined on X. We say that the function $f(\cdot)$ is *Fréchet* (*Gateaux*) *differentiable at* $x_0 \in U_i$ if the function $f(\phi_i^{-1}(\cdot))$ is Fréchet (resp. Gateaux) differentiable at $\phi_i(x_0)$. Since for every Fréchet differentiable at $\phi_i(x_0)$ function $g(\cdot)$ and any $\sigma(\cdot) \in C^{1,u}_{\mathbf{E}}$ the function $g(\sigma(\cdot))$ is Fréchet differentiable at $\sigma(\phi_i(x_0))$, the definition of Fréchet differentiability is the same for all compatible $C^{1,u}_{\mathbf{E}}$-atlases. Situation with Gateaux differentiability is not so nice. However, if we restrict ourselves to locally Lipschitz functions the situation is the same, since for every locally Lipschitz Gateaux differentiable at $\phi_i(x_0)$ function $g(\cdot)$ and any $\sigma(\cdot) \in C^{1,u}_{\mathbf{E}}$ the function $g(\sigma(\cdot))$ is Gateaux differentiable at $\sigma(\phi_i(x_0))$.

Much more difficult is a problem, how define a "convex" function. It looks that a natural definition is following: we say that a function $f(\cdot)$ defined on M is *"convex"* if $f(\phi_i^{-1}(\cdot))$ defined on $\mathbf{E}$ is locally convex. This definition has however a serious disadvantage. Namely, it is obvious that the "convexity" of the "convex functions" in this case ought be independent of the chart. In other words we ought to define a class $\mathcal{C}$ of real-valued functions $f(\cdot)$ such that the domain of $f(\cdot)$ is an open subset $\mathrm{dom} f = \Omega_f \subset \mathbf{E}$ and

$1_{\mathcal{C}}$. every locally convex function belongs to $\mathcal{C}$,

$2_{\mathcal{C}}$. if $f \in \mathcal{C}$ and $\sigma(\cdot)$ is a local diffeomorphism of Ω_f then for each $x \in \Omega_f$, there is an open set U, $x \in U \subset \Omega_f$, such that $f_U(\cdot)$ being the restriction of $f(\sigma(\cdot))$ to the set U belongs to $\mathcal{C}$,

$3_{\mathcal{C}}$. for each $f \in \mathcal{C}$, the function $f(\cdot)$ is

(a). Fréchet differentiable on a dense G_δ-set of its domain provided $\mathbf{E}$ is an Asplund space,

(b). Gateaux differentiable on dense G_δ-set of its domain provided $\mathbf{E}$ is separable.

Having the class $\mathcal{C}$ satisfying $1_{\mathcal{C}}$ and $2_{\mathcal{C}}$ and $3_{\mathcal{C}}$, we can easily to define the class of functions $\mathcal{F}(M)$ defined on manifolds and satisfying $1_{\mathcal{F}}$ and $2_{\mathcal{F}}$ and $3_{\mathcal{F}}$. Namely, we say that a function $f(\cdot)$ defined on a manifold M with an $C^{1,u}_{\mathbf{E}}$-atlas (U_i, ϕ_i) (i ranging in some indexing set) belongs to $\mathcal{F}(M)$ if for all i $f(\phi_i^{-1}(\cdot)) \in \mathcal{C}$.

The simplest example of the class $\mathcal{C}$ having properties $1_{\mathcal{C}}$ and $2_{\mathcal{C}}$ and $3_{\mathcal{C}}$ is the following class $\mathcal{C}_0$. We say that a function $f \in \mathcal{C}_0$, if for all $x \in \mathrm{dom} f$ there are an open set U, $x \in U \subset \Omega_f$, a diffeomorphism σ of U onto $\sigma(U)$ and a locally convex function $g(\cdot)$ defined on $\sigma(U)$ such that $f(\cdot) = g(\sigma(\cdot))$. It is easy to see that the class $\mathcal{C}_0$ has the requested property. In the case (b) we use the fact that locally convex function is locally Lipschitzian.

However, the class $\mathcal{C}_0$ has serious disadvantages. The first one is that there is not nice description of this class similar to local convexity, second is that the sum of two functions f, g belonging to the class $\mathcal{C}_0$ and having the same domain may not belongs to the class $\mathcal{C}_0$.

Example 1. Let $\mathbf{E} = \mathbb{R}$. Let

$$f(x) = [arctan(x-a)]^2$$

and

$$g(x) = [arctan(x+a)]^2$$

Of course the both functions $f, g \in \mathcal{C}_0$ as a composition of quadratic function and diffeomorphisms. Let a be chosen in such a way that $arctan(a) > 0.99\frac{\pi}{2}$. Thus

$$f(a) + g(a) = f(-a) + g(-a) = [arctan(2a)]^2 < (\frac{\pi}{2})^2$$

On the other hand

$$f(0) + g(0) = 2[arctan(2a)]^2 > (0.99\frac{\pi}{2})^2$$

It implies that $f(x) + g(x)$ has local strict maximum at the point 0. Thus $f(\cdot) + g(\cdot) \notin \mathcal{C}_0$, since a function belonging $\mathcal{C}_0$ does not have a local maximum.

Of course we can replace $\mathcal{C}_0$ by its cone

$$\mathcal{C}_\infty = \{f | f = \sum_{i=1}^{n} f_i(\cdot), \quad f_i \in \mathcal{C}_0\}.$$

It is easy to check that $\mathcal{C}_\infty$ has requested property, but still there is no a natural description of $\mathcal{C}_\infty$.

In the paper we propose another class of functions, which seems more proper. It will be locally strongly paraconvex functions.

Now we recall the notion of strongly $\alpha(\cdot)$-paraconvex functions (Rolewicz (2000)). Let $\alpha(\cdot)$ be a nondecreasing function mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that

$$\lim_{t \downarrow 0} \frac{\alpha(t)}{t} = 0. \tag{1}$$

Let a real-valued continuous function $f(\cdot)$ be defined on an open convex subset $\Omega \subset X$. We say that the function $f(\cdot)$ is *strongly $\alpha(\cdot)$-paraconvex* if for all $x, y \in \Omega$ and $0 \le t \le 1$ we have

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) + \min[t, (1-t)]\alpha(\|x - y\|). \tag{2}$$

The set of all strongly $\alpha(\cdot)$-paraconvex functions defined on Ω we shall denote by $\alpha PC(\Omega)$. If there is an $\alpha(\cdot)$ satisfying (1) such that a function is strongly $\alpha(\cdot)$-paraconvex we say that it is strongly paraconvex. The set of all strongly paraconvex functions defined on Ω we shall denote by $PC(\Omega)$.

Let X be a Banach spaces over reals. Let $f(\cdot)$ be a real-valued function defined on an open subset $\Omega \subset X$. We say that $f(\cdot)$ is *locally strongly paraconvex* if for all $x_0 \in \Omega$ there is a convex open neighbourhood U_{x_0} of x_0 such that the function $f(\cdot)$ restricted to U_{x_0}, $f\Big|_{U_{x_0}}(\cdot)$, is strongly paraconvex.

The set of all locally strongly paraconvex functions defined on Ω we shall denote by $PC^{Loc}(\Omega)$.

It is easy to see that the class $PC^{Loc}(\Omega)$ satisfies condition $1_{\mathcal{C}}$.

The essential role in showing that it also satisfies condition $2_{\mathcal{C}}$ is played by the following

Proposition 2. *Let Ω_X (Ω_Y) be an open convex set in a Banach space over reals X (resp. Y). Let σ be a mapping of a Ω_X into Ω_Y such that the differentials of $\partial\sigma\big|_x$ are uniformly continuous function of x in the norm topology. Then there is a function $\beta(\cdot)$ mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that*

$$\lim_{t \downarrow 0} \frac{\beta(t)}{t} = 0. \tag*{$(1)_\beta$}$$

and such that for all $x, y \in \Omega_X$ and $0 \le t \le 1$

$$\|\sigma\big(tx + (1-t)y\big) - t\sigma(x) + (1-t)\sigma(y)\| \le \min[t, (1-t)]\beta(\|x-y\|). \tag{3}$$

Proof. We shall start the proof of Proposition 2 with special case, namely when $Y = \mathbb{R}$ is one-dimensional. In other words, we consider a real valued function $f(\cdot)$ defined on be an open convex set $\Omega \subset X$. By our assumptions $f(\cdot)$ is differentiable on Ω and that the differentials of $f\big|_x$ are uniformly continuous function of x in the norm topology. In other words, there is a function β_0 mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that

$$\lim_{t \downarrow 0} \beta_0(t) = 0. \tag{4}$$

and

$$\|\partial f\big|_x - \partial f\big|_y\| \le \beta_0(\|x-y\|). \tag{5}$$

We define

$$F(t) = f\big(tx + (1-t)y\big) - [tf(x) + (1-t)f(y)].$$

It is easy to observe that $F(0) = F(1) = 0$. Now we shall calculate its derivative

$$\frac{dF}{dt}\Big|_t = \partial f\Big|_{(tx+(1-t)y)}(x-y) - f(x) + f(y). \tag{6}$$

Since $F(0) = F(1) = 0$, by the Rolle theorem there is $t_0, 0 \le t_0 \le 1$, such that $\frac{dF}{dt}\Big|_{t_0} = 0$. Thus for arbitrary $t, 0 \le t \le 1$

$$\begin{aligned} |\frac{dF}{dt}\Big|_t| = |\frac{dF}{dt}\Big|_t - \frac{dF}{dt}\Big|_{t_0}| \le |\partial f\Big|_{(tx+(1-t)y)} - \partial f\Big|_{(t_0x+(1-t_0)y)}(x-y)| \\ \le \beta_0\Big(\|(tx + (1-t)y) - (t_0x + (1-t_0)y)\|\Big)\|x-y\| \le \beta_0\Big(\|x-y\|\Big)\|x-y\| \\ = \beta\Big(\|x-y\|\Big), \end{aligned} \tag{7}$$

where the function $\beta(t) = t\beta_0(t)$ satisfies $(1)_\beta$.

Since $F(0) = F(1) = 0$, by (7) we have

$$F(t) = \int_0^t \frac{dF}{ds}\Big|_s ds \le t\beta\Big(\|x-y\|\Big)$$

and

$$F(t) = \int_t^1 \frac{dF}{ds}\Big|_s ds \le (1-t)\beta\Big(\|x-y\|\Big).$$

Therefore

$$F(t) \le \min[t, (1-t)]\beta(\|x-y\|). \tag{8}$$

Now we consider the general case.

Since the differentials of $\partial\sigma\big|_x$ are uniformly continuous function of x in the norm topology, there is a function β_0 mapping the interval $[0,+\infty)$ into the interval $[0,+\infty]$ satisfying (4) and

$$\|\partial\sigma\big|_x - \partial\sigma\big|_y\| \leq \beta_0(\|x-y\|). \tag{9}$$

Take any functional $\phi \in Y^*$ of norm one. We define

$$f_\phi(t) =: \phi\Big(\sigma\big(tx+(1-t)y\big) - \big(t\sigma(x)+(1-t)\sigma(y)\big)\Big). \tag{9}$$

Observe that the differentials of the real-valued f_ϕ, $\partial f_\phi\Big|_x$ are uniformly continuous function of x in the norm topology. Thus by Proposition 2

$$f_\phi(t) \leq \min[t,(1-t)]\beta(\|x-y\|). \tag{10}$$

Since ϕ was an arbitrary linear functional of norm one by (10) we get

$$\|\sigma\big(tx+(1-t)y\big) - t\sigma(x)+(1-t)\sigma(y)\| = \sup_{\{\phi:\|\phi\|=1\}} \phi(\sigma\big(tx+(1-t)y\big) - t\sigma(x)+(1-t)\sigma(y))$$

$$= \sup_{\{\phi:\|\phi\|=1\}} f_\phi(t) \leq \min[t,(1-t)]\beta(\|x-y\|). \tag{11}$$

□

By Proposition 2 we get

Theorem 3 (Rolewicz (2007)). *Let Ω_X (Ω_Y) be an open set in a Banach space over reals X (resp. Y). Let $f(\cdot)$ be a real-valued locally strongly paraconvex function defined on Ω_Y. Let σ be a mapping of a Ω_X into Ω_Y such that the differentials of $\sigma\big|_x$ are locally uniformly continuous function of x in the norm topology. Then the composed function $f(\sigma(\cdot))$ is locally strongly paraconvex.*

Proof. Let $x_0 \in \Omega_X$. Since $f(\cdot)$ is a real-valued locally strongly paraconvex function, there are an open convex neighborhood of $\sigma(x_0)$ $U_{\sigma(x_0)} \subset \Omega_Y$ ans a nondecreasing function $\alpha_U(\cdot)$ satisfying (1) such that for all $x, y \in U_{\sigma(x_0)}$ and $0 \leq t \leq 1$

$$f\big(tx+(1-t)y\big) \leq tf(x)+(1-t)f(y)+\min[t,(1-t)]\alpha_U(\|x-y\|). \tag{12}$$

Recall that $f(\cdot)$ restricted to $U_{\sigma(x_0)}$ is a Lipschitz function (Rolewicz (2000)). We shall denote the corresponding Lipschitz constant by M. Thus by Proposition 1

$$|f\Big(\sigma\big(tx+(1-t)y\big)\Big) - f\Big(t\sigma(x)+(1-t)\sigma(y)\Big)|$$

$$\leq M\|\sigma\big(tx+(1-t)y\big) - t\sigma(x)+(1-t)\sigma(y)\| \leq M\min[t,(1-t)]\beta(\|x-y\|). \tag{13}$$

Therefore

$$f\Big(\sigma\big(tx+(1-t)y\big)\Big) \leq f\Big(t\sigma(x)+(1-t)\sigma(y)\Big) + M\min[t,(1-t)]\beta(\|x-y\|)$$

$$\leq tf(\sigma(x))+(1-t)f(\sigma(y))+\min[t,(1-t)]\alpha(\|\sigma(x)-\sigma(y)\|)+M\min[t,(1-t)]\beta(\|x-y\|)$$

$$=tf(\sigma(x))+(1-t)f(\sigma(y))+\min[t,(1-t)]\Bigg(\alpha(\|\sigma(x)-\sigma(y)\|)+\beta(\|x-y\|)\Bigg). \qquad (14)$$

Since $\sigma(\cdot)$ is locally uniformly differentiable, it is also locally Lipschitz, i.e. there are a neighbourhood V_{x_0} of x_0 and a constant N such that for $x, y \in V_{x_0}$

$$\|\sigma(x)-\sigma(y)\| \leq N\|x-y\|. \qquad (15)$$

Let

$$\gamma(t)=\alpha(Nt)+\beta(t). \qquad (16).$$

It is easy to check that $\gamma(\cdot)$ satisfies (1). Moreover by (14) and (15) the function $f(\sigma(\cdot))$ is strongly $\gamma(\cdot)$-paraconvex on V_{x_0}. Therefore is locally strongly paraconvex. □

Condition $3_{\mathcal{C}}$ is an immediate consequence of

Theorem 4. (Rolewicz (1999), (2001), (2001b), (2002), (2005), (2005b), (2006), Zajíček (2007)) *Let Ω_X be an open set in a Banach space over reals X. Let $f(\cdot)$ be a real-valued strongly paraconvex function defined on Ω_X. Then the function $f(\cdot)$ is:*

(a). *Fréchet differentiable on a dense G_δ-set provided X is an Asplund space,*

(b). *Gateaux differentiable on dense G_δ-set provided X is separable.*

By Michael theorem (Michael (1953)) we immediately obtain a local version of Theorem 4. Namely we have

Theorem 5. (Rolewicz,(2007)) *Let Ω_X be an open set in a Banach space over reals X. Let $f(\cdot)$ be a real-values locally strongly paraconvex function defined on Ω_X. Then the function $f(\cdot)$ is:*

(a). *Fréchet differentiable on a dense G_δ-set provided X is an Asplund space,*

(b). *Gateaux differentiable on dense G_δ-set provided X is separable.*

Combining Theorems 3 and 5 we trivially get

Theorem 6. (Rolewicz (2007)) *Let Ω_X (Ω_Y) be an open set in a Banach space over reals X (resp. Y). Let $f(\cdot)$ be a real-valued locally strongly paraconvex function defined on Ω_Y. Let σ be a mapping of a Ω_X into Ω_Y such that the differentials of $\sigma\big|_x$ are locally uniformly continuous function of x. Then the composed function $f(\sigma(\cdot))$ is:*

(a). *Fréchet differentiable on a dense G_δ-set provided X is an Asplund space,*

(b). *Gateaux differentiable on dense G_δ-set provided X is separable.*

We say that a real-valued function $f(\cdot)$ defined on a $C^{1,u}_{\mathbf{E}}$-manifold M is locally strongly paraconvex on M if there is a $C^{1,u}_{\mathbf{E}}$-atlas (U_i, ϕ_i) such that for all i the function $f(\phi_i^{-1}(\cdot))$ locally strongly paraconvex on the set $\phi_i(U_i) \subset \mathbf{E}$.

Basing on Theorem 6 and the definitions of differentiability of functions on manifold we immediately obtain

Theorem 7 (Rolewicz (2007)). *Let M be a $C^{1,u}_{\mathbf{E}}$-manifold. Let $f(\cdot)$ be a real-valued locally strongly paraconvex function defined on M. Then it is:*

(a). *Fréchet differentiable on a dense G_δ-set provided $\mathbf{E}$ is an Asplund space,*

(b). *Gateaux differentiable on a dense G_δ-set provided $\mathbf{E}$ is separable.*

Now we shall determine $C^{1,u}_{\mathbf{E}}$-submanifold in the classical way (compare Lang (1962)).

Let M be a $C^{1,u}_{\mathbf{E}}$-manifold. Let N be a subset of M. We assume that for each point $y \in N$ there exist a chart (V, ψ) in M such that $V_1 = \psi(V \cap N)$ is an open in some Banach subspace $\mathbf{E}_1 \subset \mathbf{E}$. The map ψ induces a bijection

$$\psi_1 : Y \cap V \to V_1 \tag{17}$$

and moreover $\psi_1 \in C^{1,u}_{\mathbf{E}_1}$

The collection of pairs $(N \cap V, \psi_1)$ obtained in the above manner constitute the atlas for N. We shall call N $C^{1,u}_{\mathbf{E}_1}$*-submanifold* of M.

Theorem 8 (Rolewicz (2007)). *Let M be a $C^{1,u}_{\mathbf{E}}$-manifold. Let N be an its $C^{1,u}_{\mathbf{E}_1}$-submanifold. Let $f(\cdot)$ be a real-valued locally strongly paraconvex function defined on M. Then the restriction $f\Big|_N$ is locally strongly paraconvex function defined on N.*

Corollary 9. *Let $f(\cdot)$ be a convex function defined on $\mathbb{R}^n$. Let M be and m-dimensional manifold, $m < n$, imbedded in $\mathbb{R}^n$. Then the restriction of the function $f(\cdot)$ to M is differentiable on a dense G_δ-set.*

REFERENCES.

Asplund, E.: (1966), *Farthest points in reflexive locally uniformly rotund Banach spaces*, Israel Jour. Math., **4**, pp. 213 - 216.

Asplund, E.: (1968), *Fréchet differentiability of convex functions*, Acta Math., **121**, pp. 31 - 47.

Lang, S.: (1962), **Introduction to differentiable manifolds**, Interscience Publishers (division of John Wiley & Sons) New York, London.

Mazur, S.: (1933), *Über konvexe Mengen in linearen normierten Räumen*, Stud. Math., **4**, pp. 70 - 84.

Michael,E.: (1954), *Local properties of topological spaces*, Duke Math. Jour. **21**, pp. 163 - 174.

Phelps, R.R.: (1989), **Convex Functions, Monotone Operators and Differentiability**, Lecture Notes in Mathematics, Springer-Verlag **1364**.

PREISS, D.,ZAJÍČEK, L.: (1984), *Stronger estimates of smallness of sets of Fréchet nondifferentiability of convex functions*, Proc. 11-th Winter School, Suppl. Rend. Circ. Mat di Palermo, ser II, **3**, pp. 219 - 223.

ROLEWICZ, S.: (1999), *On $\alpha(\cdot)$-monotone multifunction and differentiability of γ-paraconvex functions*, Stud. Math. **133**, pp. 29 - 37.

ROLEWICZ, S.: (2000), *On $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions*, Control and Cybernetics **29**, pp. 367 - 377.

ROLEWICZ, S.: (2001), *On the coincidence of some subdifferentials in the class of $\alpha(\cdot)$-paraconvex functions*, Optimization **50**, pp. 353 - 360.

ROLEWICZ, S.: (2001b), *On uniformly approximate convex and strongly $\alpha(\cdot)$-paraconvex functions*, Control and Cybernetics **30**, pp. 323 - 330.

ROLEWICZ, S.: (2002), *$\alpha(\cdot)$-monotone multifunctions and differentiability of strongly $\alpha(\cdot)$-paraconvex functions*, Control and Cybernetics **31**, pp. 601 - 619.

ROLEWICZ, S.: (2005), *On differentiability of strongly $\alpha(\cdot)$-paraconvex functions in non-separable Asplund spaces*, Studia Math. **167**, pp. 235 - 244.

ROLEWICZ, S.: (2005b), *Paraconvex Analysis*, Control and Cybernetics **34**, 951 - 965.

ROLEWICZ, S.: (2006), *An extension of Mazur Theorem about Gateaux differentiability*, Studia Math. **172**, pp. 243 - 248.

ROLEWICZ, S.: (2007), *Paraconvex Analysis on $C^{1,u}_{\mathbf{E}}$-manifolds*, Optimization **56**, pp. 49 - 60