INSTITUTE OF MATHEMATICS
of the
Polish Academy of Sciences

ul. Śniadeckich 8, P.O.B. 21, 00-956 Warszawa 10, Poland http://www.impan.pl

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Piotr Miłoś

A note on small branching fluctuation limits of catalytic superprocesses with immigration

Presented by Jerzy Zabczyk

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A note on small branching fluctuation limits of catalytic superprocesses with immigration *

Piotr Miłoś

Institute of Mathematics, Polish Academy of Sciences. Śniadeckich 8, 00-956 Warsaw

Abstract

We establish a small branching fluctuation limit theorem for a class of catalytic superprocesses with immigration. The limit process, under some additional conditions, can be represented as an $\mathcal{L}^2$-valued Ornstein-Uhlenbeck process. This note aims to extend results of [2] to larger class of superprocesses.

AMS subject classification: primary 60F17, 60G20, secondary 60G15

Key words: Superprocesses; small fluctuations limit; Ornstein-Uhlenbeck process; catalytic superprocess, superprocess with immigration

1 Introduction

In [2] the authors consider a small branching fluctuation limits of a catalytic superprocess with immigration. Their assumption that the "particle motion process" is absorbing barrier Brownian motion (ABM), seems to be restrictive. In the note we extend their results to a broader class of superprocesses, assuming that the "motion process" can be any symmetric α-stable Lévy process. In the case of $\alpha \in (0, 1]$ additional, technical assumptions were needed in the definition of the catalyst. It is not quite clear if they can be relaxed. The assumptions we put on the branching mechanism and immigration are not restrictive. More details and formal definition of the superprocesses is in Section 2.

A sequence of such superprocesses converges to a deterministic process X when the branching is being suppressed (what is more the limit process is constant if $X_0 = \lambda$, where λ is the Lebesgue measure i.e. the invariant measure of α-stable motion). In order to make the presentation more comprehensive we explain this convergence in an intuitive (though not quite formal) way. Recall, that a superprocess arises as a limit of branching particle systems by increasing the initial number of particles (and changing other parameters of the systems appropriately). Let us consider now one of this branching systems. It converges to a system consisting of independent

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particles moving according to the α-stable Lévy motion when the branching is being suppressed. In turn such a system gives rise to the deterministic superprocesses X. We want to study the properties of the convergence to X in terms of a central limit theorem. The studies of this kind of convergence, called also small branching fluctuations, were initiated in [6]. Typically the limit process is an Ornstein-Uhlenbeck. In our case under additional assumptions the limit process was proved to be an $\mathcal{L}^2$-valued Ornstein-Uhlenbeck process with Lévy noise. This form can be explained intuitively by competition of two antagonistic forces - particles motion vs. branching (see also Remark 3.1).

The note is organised as follows. In Section 2 we describe the branching particle system formally. In Section 3 we give results. Next, in Section 4 we present results. Finally, at the end of the note we pose some open questions and propose possible extensions.

In the proofs we basically follow the lines of proofs in [2] resolving some technical issues arising in our more general case.

Notation

$\mathcal{M}(\mathbb{R}^d)$ - space of finite Borel measures on $\mathbb{R}^d$ endowed with the topology of weak convergence,

$B^+(\mathbb{R}^d)$ - bounded positive Borel functions,

$\langle \nu, f\rangle, \nu(f)$ - $\int f(x)\nu(\mathrm{d}x)$,

λ - the Lebesgue measure,

$\Rightarrow_{fdd}$ - weak finite-dimensional convergence

$\mathcal{S}'(\mathbb{R}^d)$ - space of tempered distribution (dual to the space of Schwartz functions)

2 Catalytic superprocess with immigration

In this section we describe the "ingredients" of the class of superprocesses we study. Let us note that the class is quite broad in fact only assumptions imposed on the motion are really restrictive. In each paragraph we comment on the assumptions but the reader is also encouraged to read the questions at the end of note to see possible extensions.

Motion The "motion" of the superprocess will be given by a symmetric α-stable Lévy motion, $\alpha \in (0, 2]$, denoted by $(\vartheta_t)_t$. This is the major extensions compared to [2] and [6]. On the one hand this give a large class of superprocesses on the other the α-stable Lévy motion has still good properties which allow analytical treatment. What is more this allows other assumptions to be quite general and yields a result, which seems to be elegant.

We denote the semigroup and the transition density of a symmetric α-stable Lévy motion by $(P_t)_{t\geq 0}$ and $(p_t)_{t\geq 0}$ respectively. Moreover we denote its Riesz kernel by

R. It is well known that

$$R(x)=\Gamma\left(\frac{d-\alpha}{2}\right)\left(2^{\alpha}\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{\alpha}\right)\right)^{-1}|x|^{\alpha-d},$$

and slightly abusing notation by R we denote also Riesz potential

$$R\mu(x)=\int_{\mathbb{R}^d}R(x-y)\mu(\mathrm{d}y).$$

We have

$$R\mu(x)=\int_0^{+\infty}P_s^*\mu(x)\mathrm{d}s.$$

Branching law Consider a function

$$\Phi(x,z)=c(x)z^2+\int_0^{+\infty}\left(e^{-zu}-1+zu\right)m(x,\mathrm{d}u), \tag{1}$$

where c is a bounded, positive measurable function and m is a measure such that $u^2m(x,\mathrm{d}u)$ is a bounded kernel (ie. $x\to\int u^2m(x,\mathrm{d}u)$) is bounded. This function corresponds to the generating function of branching law of "underlying" branching particle system. This form is very general covering almost any possible branching law. Note also that it admits the dependence on the space.

Catalysis First papers concerning catalysis appeared in early nineties (e.g. [1]). Intuitively it models interaction of the particles of the system with a catalyst particles which can be concentrated on a "small set". Mathematically a concentration of the catalyst is described by a measure which affects a branching intensity.
The branching intensity is determined by a positive additive functional L. It is well known fact that the functionals are in one-to-one (Revuz) correspondence with measures (under some technical assumptions). Denote by η the measure corresponding to L. Roughly speaking η defines how the branching intensity varies in space. The most interesting case is when there are points (or set of 0 Lebesgue measure) with positive mass
We consider two classes

- when $\alpha\in(0,1]$ and $d=1$ or $\alpha\in(0,2)$ and $d>1$. By [5] we know that η have to fulfil

$$\sup_x R\eta(x)<+\infty.$$

  Admissibility We also assume that $R\eta(x)$ is continuous and η can be decomposed $\eta=\eta_1$ such that $R\eta_1(x)$ converges to 0 as $x\to+\infty$ and $R\eta_2(x)$ is periodic.

- when $\alpha\in(1,2]$ η and $d=1$ L can be represented by

$$L_t=\langle\eta,l_t\rangle,$$

  where $(l_t)_{t\geq 0}$ denotes a local time of ϑ and measure η fulfilling
  Admissibility We assume that

$$\eta([x,x+l])\leq cl,\forall_x,$$

  for positive constants c,l.

The second case is particularly illustrative, by putting $\eta = \delta_x$ we make the branching be possible only when a particle is in x which corresponds to a particle of catalyst in this point.
Aside for the *admissibility* conditions, the assumptions imposed in this section are very mild. They are just assumptions required to existence of an additive functional L. The assumptions of *admissibility* is not very restrictive but can be potentially weakened, this issue will be addressed in Section 4 too. Now we can define a catalytic superprocess

Proposition 2.1. *There exists a measure-valued (ie. $\mathcal{M}(\mathbb{R}^d)$) process Markov process $(X_t)_{t\geq 0}$ with the Laplace functional*

$$\mathbb{E}_\mu e^{-\langle X_t,\varphi\rangle} = \exp\left\{-\mu(V_t\varphi)\right\}, \quad \varphi \in B^+(\mathbb{R}^d),$$

where V_t is a non-linear semigroup satisfying equation

$$V_t\varphi(x) = P_t\varphi(x) - \int_0^s \int_{\mathbb{R}^d} \Phi(y, V_s(y)) p_{t-s}(x-y)\eta(\mathrm{d}y)\mathrm{d}s \quad \varphi \in B^+(\mathbb{R}^d). \tag{2}$$

We will refer to this as a $(P, \eta, \Phi)-$*superprocess.*

Immigration Now let γ be a purely excessive measure for P_t (ie. $P_t^*\gamma < \gamma$). Hence there exist an (unique) entrance law $(\kappa_t)_{t>0}$ (ie. $P_s^*\kappa_t = \kappa_{s+t}$) such that

$$\int_0^{+\infty} \kappa_t \mathrm{d}t = \gamma. \tag{3}$$

Later we assume also that the κ_t has density k_t in respect to Lebesgue measure. The assumption of the excessiveness is natural it just says that the entrance law is consistent with the motion process ϑ. The assumption of the continuity of κ with respect to the Lebesgue measure is technical and is likely to be removed in future. Consider now a (P, η, Φ)-superprocess. It is known (see e.g. [8]) that κ can be lifted to a entrance law K for superprocess which is defined by the Laplace functional

$$\int_{\mathcal{M}(\mathbb{R}^d)} e^{-\langle \nu, f\rangle} K_t(\mathrm{d}\nu) = \exp\left\{-S_t(\kappa, f)\right\}, \quad f \in B^+(\mathbb{R}^d),$$

where

$$S(\kappa, f) = \kappa_t(f) - \int_0^t \int_{Rd} \Phi(y, V_s f(y))\kappa_{t-s}(y)\eta(\mathrm{d}y). \tag{4}$$

Now we are ready to define a catalytic superprocess with immigration

Proposition 2.2. *There exists a measure-valued (ie. $\mathcal{M}(\mathbb{R}^d)$) process Markov process $(X_t)_{t\geq 0}$ with the Laplace functional*

$$\mathbb{E}_\mu e^{-\langle X_t,\varphi\rangle} = \exp\left\{-\mu(V_t\varphi) - \int_0^t S_r(\kappa, f)\mathrm{d}r\right\}, \quad \varphi \in B^+(\mathbb{R}^d).$$

This process will be referred to it as a (P, L, Φ, κ)-*immigration superprocess.*

3 Results

Small branching fluctuations

Define

$$\Phi_\theta(x,z) = \Phi(x,\theta z), \quad \theta > 0. \tag{5}$$

Roughly speaking this corresponds to changing intensity and size of branching. Let γ be excessive measure for P. Let $(b_\theta)_\theta$ be a sequence Borel functions on $\mathbb{R}^d$ such that $b_\theta > c_\theta > 0$, $c_\theta \in \mathbb{R}$ and $b_\theta \to 0$ uniformly. Define semigroup P^θ by

$$P^\theta f(x) = \mathbb{E}_x f(\vartheta_t) \exp\left\{ - \int_0^t b_\theta(\vartheta_s)\mathrm{d}s \right\}.$$

It is easy to check that γ is purely excessive with respect to each P^θ hence by (3) we have an entrance law κ^θ. Finally by $(X_t^\theta)_{t\geq 0}$ we denote the $(P^\theta, L, \Phi_\theta, \kappa^\theta)$-immigration superprocess, with $X_0^\theta = \gamma$.

Note that in the case when γ is purely excessive for P_t there is no need to define P^θ and one can simply consider $(P, L, \Phi_\theta, \kappa)$ superprocess. This case has obvious intuitive meaning since it corresponds to suppressing the branching. It can be checked that

$$X^\theta \to X, \qquad \text{as } \theta \to 0, \tag{6}$$

where X is a deterministic process $X_t = \gamma$. We want to determine the speed of this convergence in terms of a central limit theorem. In order to do this we define the *fluctuations process*

$$Y^\theta = \frac{1}{\theta}\left(X^\theta - \gamma \right). \tag{7}$$

The main result of the paper identifies the limit of Y^θ

Theorem 3.1. *Let Y^θ be the fluctuation process* (7)*. Under assumptions from Section 2 the following convergence holds*

$$Y^\theta \Rightarrow_{fdd} Y, \qquad \text{as } \theta \to 0, \tag{8}$$

where Y is a $\mathcal{S}'(\mathbb{R}^d)$-valued Markov process with semigroup R_t such that

$$\int_{\mathcal{S}'(\mathbb{R}^d)} e^{-\nu(f)} R_t(\mu,\nu) = \exp\left\{ -\mu(P_t f) + \int_0^t \eta(\Phi(P_s f))ds \right\}, \quad f \in \mathcal{S}(\mathbb{R}^d)^+. \tag{9}$$

and $\Rightarrow_{fdd}$ is in a sense of the topology of $\mathcal{S}'(\mathbb{R}^d)$.

What is more, under additional assumptions the limit process Y has a regular realisation.

Theorem 3.2. *Assume $d = 1, \alpha > 1$.*
Suppose that c in (1) *is bounded and η defining catalysis is finite. Let W be a white random noise on $[0,+\infty) \times \mathbb{R}^d$ with covariance $2c(x)\mathrm{d}s\eta(\mathrm{d}x)$ and N be a compensated Poisson random field on $[0,+\infty) \times \mathbb{R}^0 \times \mathbb{R}^d$ with the intensity measure $\mathrm{d}s m(x,\mathrm{d}u)\eta(\mathrm{d}x)$. Moreover, assume that W, N are independent. Then the process*

$$Z_t(y) := \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) W(\mathrm{d}s,\mathrm{d}x) + \int_0^t \int_{\mathbb{R}^0} \int_{\mathbb{R}^d} u p_{t-s}(y-x) N(\mathrm{d}s,\mathrm{d}u,\mathrm{d}x), \tag{10}$$

is well-defined in the $\mathcal{L}^2(\Omega\times\mathbb{R}^d, \mathbb{P}\times\lambda)$ sense. What is more it is an $\mathcal{L}^2(\mathbb{R}^d,\lambda)$-valued Markov process in with semigroup

$$\int_{\mathcal{L}^2(\mathbb{R}^d,\lambda)} e^{i\nu(f)} R_t(\mu,\nu) = \exp\left\{ i\mu(P_t f) + \int_0^t \eta(\Phi(iP_s f))\mathrm{d}s \right\}, \quad f \in \mathcal{L}^2(\mathbb{R}^d,\lambda). \tag{11}$$

Remark 3.1. Informally, the process $(Z_t)_{t\geq 0}$ fulfils an equation

$$Z_t = \Delta_\alpha Z_t + \mathrm{d}L_t, \tag{12}$$

where Δ_α is the infinitesimal operator of a symmetric Lévy α-stable motion and $\mathrm{d}L = \mathrm{d}W + \mathrm{d}N$ is an $\mathcal{L}^2$-valued random noise. Hence Z can be regarded as an $\mathcal{L}^2$-valued Ornstein-Uhlenbeck process.

We can interpret Z in terms of the limiting superprocesses (recall Theorem 3.1). Process Z can be viewed as a result of the struggle of two antagonistic forces. One is the motion, which "attracts" Z towards the invariant measure (Δ_α is its infinitesimal semigroup of α-stable motion). The other is branching, which contributes random Lévy noise dL and "repels" Z from the invariant measure.

Remark 3.2. Assumptions $\alpha > 1$ and $d = 1$ are sharp in a sense that without them the integral (10) is not well defined.

4 Proofs

The proofs of Proposition 2.1 and Proposition 2.2 are technical. They include checking that the objects in Section 2 are well defined. The existence of the additive functional defining the catalysis follows from the theorems in [5]. The main tool used in the proof of Proposition 2.1 is [4, Theorem 4.1]. Its application requires checking number of technical conditions imposed on the "ingredients" of a superprocess. In Section 2 we imposed *admissibility* conditions on measure η, using them one can check that the additive functional L is admissible in a sense of the definition in Section 3.3 of [4]. Although these conditions can potentially be weakened there is not much hope of obtaining "necessary" conditions in more elegant form. Other conditions of [4, Theorem 4.1] are rather straightforward and we skip them. The existence of the process announced in Proposition 2.2 follows from discussion in [8] and "lifting techniques" presented in [3].

Convergence Now we concentrate on the proof of Theorem 3.1. Simple calculations using equations (4), (3) yield

$$\int_0^t S_r(\kappa, f)\mathrm{d}r = \gamma(f - V_t f) - \int_0^t \eta(\Phi(V_s f))\mathrm{d}s.$$

Y^θ defined by (7) are signed-measure-values processes but from now on we will treat them as $\mathcal{S}'(\mathbb{R}^d)$-valued. Consider Y^1, t is easy to check that it is a Markov process with transition semigroup $(T_t^1)_{t\geq 0}$ given by the Laplace functional

$$\int_{\mathcal{S}'(\mathbb{R}^d)} e^{\nu(f)} T_t^1(\mu, \mathrm{d}\nu) = \exp\left\{ -\mu(V_t) + \int_0^t \Phi(y, V_s f(y))\eta(\mathrm{d}y)\mathrm{d}s \right\}, \ f \in \mathcal{S}(\mathbb{R}^d)^+.$$

Similarly, it is easy to check that each Y^θ is a Markov process too. Its semigroup $(T^\theta)_{t>0}$ is given by the Laplace functional (recall also (5))

$$\int_{\mathcal{S}'(\mathbb{R}^d)} e^{\nu(f)} T_t^\theta(\mu, \mathrm{d}\nu) = \exp\left\{-\mu(\theta V_t^\theta) + \int_0^t \Phi(y, \theta V_s^\theta f(y))\eta(\mathrm{d}y)\mathrm{d}s\right\}, f \in \mathcal{S}(\mathbb{R}^d)^+, \tag{13}$$

where $(V_t^\theta)_{t\geq 0}$ is defined by (recall also (2))

$$V_t^\theta \varphi(x) = P_t\varphi(x) - \int_0^s \int_{\mathbb{R}^d} \Phi_\theta(y, V_s^\theta(y)) p_{t-s}(x-y)\eta(\mathrm{d}y)\mathrm{d}s, \quad \varphi \in B^+(\mathbb{R}^d).$$

Fix $0 = t_0 < t_1 < t_2 < \ldots < t_n$, and $f_i \in \mathcal{S}(\mathbb{R}^d)^+$. Now using Markov property and (13) and we can calculate the Laplace functional of the finite-dimensional distributions $(Y_{t_1}^\theta, Y_{t_2}^\theta, \ldots, Y_{t_n}^\theta)$, namely

$$\mathbb{E}\exp\left\{-\sum_{k=1}^n \left\langle Y_{t_k}^\theta, f_k\right\rangle\right\} = \exp\left\{\sum_{k=1}^n \int_0^{t_k - t_{k-1}} \Psi(y, \theta V_s^\theta(h_k^\theta/\theta)(y)\eta(\mathrm{d}y)\mathrm{d}s\right\},$$

where h_j^θ are defined inductively

$$h_k^\theta := f_k + \theta V_{t_{k+1}-t_k}(h_{k+1}^\theta/\theta), \quad k \in \{1, \ldots, n-1\}, \tag{14}$$

$$h_n^\theta := f_n.$$

A simple lemma (see [2, Lemma 4.1]) holds

Lemma 4.1. *If $f_\theta \to f \in B(\mathbb{R}^d)^+$ boundly as $\theta \to 0$ then*

$$\theta V_t^\theta(f_\theta/\theta) \to P_t f, \quad \textit{boundly as } \theta \to 0. \tag{15}$$

By this lemma one can easily prove that $h_j^\theta \to h_j$ as $\theta \to 0$, where h_j are defined as follows

$$h_k := P_{t_{k+1}-t_k} h_{k+1}, \quad k \in \{1, \ldots, n-1\},$$

$$h_n := f_n.$$

Finally, we can write

$$\lim_{\theta\to 0} \mathbb{E}\exp\left\{-\sum_{k=1}^n \left\langle Y_{t_k}^\theta, f_k\right\rangle\right\} = \exp\left\{\sum_{k=1}^n \int_0^{t_k - t_{k-1}} \Psi(y, P_s h_k(y))\eta(\mathrm{d}y)\mathrm{d}s\right\}$$

This, following the lines of reasoning in [7], proves convergence announced in the Theorem 3.1.

Representation Now we proceed to the proof of Theorem 3.2, recall definitions of the random objects there. By properties of the integration with respect to a Gaussian random fields we have

$$\int_{\mathbb{R}} \mathbb{E}\left(\int_0^t \int_{\mathbb{R}} p_{t-s}(y-x) W(\mathrm{d}s, \mathrm{d}x)\right)^2 \mathrm{d}y = \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} p_{t-s}^2(y-x)\mathrm{d}s\eta(\mathrm{d}x)\mathrm{d}y =$$

Applying Fubini's theorem and inequality $\|p_s\|_2^2 \leq ct^{-1/\alpha}$ (see eg. [9, (3.29)]) one gets

$$\int_{\mathbb{R}} \int_0^t \|p_s\|_2^2 \mathrm{d}s\eta(\mathrm{d}x) \leq \int_0^t s^{-1/\alpha}\mathrm{d}s \int_{\mathbb{R}^d} c(x)\eta(\mathrm{d}x) < +\infty$$

Analogously by properties of the integration with respect to a (compensated) Poisson random field we have

$$\int_{\mathbb{R}} \mathbb{E}\left(\int_0^t \int_{\mathbb{R}^0} \int_{\mathbb{R}^d} u p_{t-s}(y-x) N(\mathrm{d}s, \mathrm{d}u, \mathrm{d}x)\right)^2 \mathrm{d}y =$$

$$\int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}^0} \int_{\mathbb{R}^d} u^2 p_{t-s}^2 (y-x) m(x, \mathrm{d}u) \eta(\mathrm{d}x) \mathrm{d}y = \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}^0} u^2 \|p_s\|_2^2 m(x, \mathrm{d}u) \eta(\mathrm{d}x) \leq$$

Using inequality $\|p_s\|_2^2 \leq ct^{-1/\alpha}$ again, we get

$$\int_0^s s^{-1/\alpha} \mathrm{d}s \int_{\mathbb{R}} \int_{\mathbb{R}^0} u^2 m(x, \mathrm{d}u) \eta(\mathrm{d}x) < +\infty$$

This taking into account the independence of N and W proves that Z_t lays in $\mathcal{L}^2(\Omega \times \mathbb{R}^d, \mathbb{P} \times \lambda)$. It is easy to check that (11) defines a semigroup on bounded functions on $\mathcal{L}^2(\mathbb{R}^d, \lambda)$ hence it generates a Markov family $(\mathcal{T}_t g + Y_t)_{T \geq 0}$. The last step of the theorem is to show that the finite-dimensional distributions of $(Y_t)_{t \geq 0}$ and $(Z_t)_{t \geq 0}$ are equal. As an example we will prove that this in case of two dimensional-distributions. Let $t_1 < t_2$ taking into account the fact that N and W are independent one can write

$$\mathbb{E} \exp\left\{-i\left(\langle Z_{t_1}, f_1\rangle + \langle Z_{t_2}, f_2\rangle\right)\right\} = A(f_1, f_2) B(f_1, f_2)$$

where

$$A(f_1, f_2) = \mathbb{E} \exp\left\{-i \left(\int_0^{t_1} \int_{\mathbb{R}^d} \mathcal{T}_{t_1-s} f_1(x) W(\mathrm{d}s, \mathrm{d}x) + \int_0^{t_2} \int_{\mathbb{R}^d} \mathcal{T}_{t_2-s} f_1(x) W(\mathrm{d}s, \mathrm{d}x)\right)\right\}$$

and

$$B(f_1, f_2) = \mathbb{E} \exp\left\{-i \left(\int_0^{t_1} \int_{\mathbb{R}^0} \int_{\mathbb{R}^d} u \mathcal{T}_{t_1-s} f(x) N(\mathrm{d}s, \mathrm{d}u, \mathrm{d}x) + + \int_0^{t_2} \int_{\mathbb{R}^0} \int_{\mathbb{R}^d} u \mathcal{T}_{t_2-s} f(x) N(\mathrm{d}s, \mathrm{d}u, \mathrm{d}x)\right)\right\}$$

Direct calculations yield

$$A(f_1, f_2) = \mathbb{E} \exp\left\{-i \left(\int_0^{t_1} \int_{\mathbb{R}^d} \mathcal{T}_{t_1-s}\left(f_1 + \mathcal{T}_{t_2-t_1} f_2\right)(x) W(\mathrm{d}s, \mathrm{d}x) + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \mathcal{T}_{t_2-s} f_1(x) W(\mathrm{d}s, \mathrm{d}x)\right)\right\}$$

The integrals on disjoint intervals are independent hence

$$A(f_1, f_2) = \mathbb{E} \exp\left\{-i\left(\int_0^{t_1} \int_{\mathbb{R}^d} \mathcal{T}_{t_1-s}\left(f_1 + \mathcal{T}_{t_2-t_1} f_2\right)(x) W(\mathrm{d}s, \mathrm{d}x)\right)\right\}$$
$$\mathbb{E} \exp\left\{-i\left(\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \mathcal{T}_{t_2-s} f_1(x) W(\mathrm{d}s, \mathrm{d}x)\right)\right\} =$$

By the theory of integration with respect to a Gaussian random field we get

$$A(f_1, f_2) = \exp\left\{-\int_0^{t_1}\int_{\mathbb{R}^d} \left[\mathcal{T}_{t_1-s}\left(f_1 + \mathcal{T}_{t_2-t_1} f_2\right)\right]^2(x)c(x)\eta(\mathrm{d}x)\right\}$$
$$\exp\left\{-\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \left[\mathcal{T}_{t_2-s} f_2\right]^2(x)c(x)\eta(\mathrm{d}x)\right\}$$

Hence

$$A(f_1, f_2) = \exp\left\{-\int_0^{t_1}\int_{\mathbb{R}^d} \left[\mathcal{T}_{t_1-s} f_1\right]^2(x)c(x)\eta(\mathrm{d}x)\right\}$$
$$\exp\left\{-\int_0^{t_2}\int_{\mathbb{R}^d} \left[\mathcal{T}_{t_2-s} f_2\right]^2(x)c(x)\eta(\mathrm{d}x)\right\}$$

$B(f_1, f_2)$ can be treated analogously, we get

$$B(f_1, f_2) =$$
$$\exp\left\{-\int_0^{t_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[\exp(iu\mathcal{T}_{t_1-s} f_1(x) - 1 - iu\mathcal{T}_{t_1-s} f_1(x))c(x)\eta(\mathrm{d}x)m(x, \mathrm{d}u)\right]\right\}$$
$$\exp\left\{-\int_0^{t_2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[\exp(iu\mathcal{T}_{t_2-s} f_2(x) - 1 - iu\mathcal{T}_{t_2-s} f_2(x))c(x)\eta(\mathrm{d}x)m(x, \mathrm{d}u)\right]\right\}$$

Finally

$$\mathbb{E}\exp\left\{-i\left(\langle Z_{t_1}, f_1\rangle + \langle Z_{t_2}, f_2\rangle\right)\right\} =$$
$$\exp\left\{\int_0^{t_1}\int_{\mathbb{R}^d} \phi(\mathcal{T}_{t_1-s} f_1(x))\mathrm{d}x\mathrm{d}s + \int_0^{t_2}\int_{\mathbb{R}^d} \phi(\mathcal{T}_{t_2-s} f_2(x))\mathrm{d}x\mathrm{d}s\right\}$$

where $\phi(x, z) = \Phi(x, iz)$,recall (1), namely $\psi(x, z) = -c(x)z^2 + \int_0^{+\infty}\left(e^{-izu} - 1 + zu\right)m(x, \mathrm{d}u)$. Now it is enough to check that this the Markov process generated by $(R_t)_{t\geq 0}$ (recall (11)) has the same two-dimensional distributions. The same argument can be applied in order to check that n-dimensional distributions coincide.

5 Further questions

The problem exhibited in the paper still may be the field of further studies. The assumptions imposed on the system can be relaxed in various ways, as described in Section 2. Further in Theorem 3.1 the question of convergence in a functional space (e.g. Skorohod space $\mathcal{D}([0, 1], \mathcal{S}'(\mathbb{R}^d))$) arises. Remark 3.2 explains why condition $\alpha > 1$, $d = 1$ is sharp but it is possible to extend the representation of Theorem 3.2 to a space larger than $\mathcal{L}^2$.

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