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CONDITIONAL STABILITY ESTIMATES AND REGULARIZATION WITH APPLICATIONS TO CAUCHY PROBLEMS FOR THE HELMHOLTZ EQUATION

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Abstract. In this paper we consider the problem of reconstructing solutions $x^\dagger$ of ill-posed problems $Ax = y$ where A is a linear operator between Hilbert spaces X and Y. We assume that instead of exact data some noisy data $y^\delta \in Y$ with $\|y - y^\delta\| \le \delta$ are given and that $x^\dagger$ possesses a certain solution smoothness which we describe by $x^\dagger \in M$ with some source set $M \subset X$. We discuss the following questions: (i) Which best possible accuracy can be obtained for identifying $x^\dagger$ from noisy data $y^\delta \in Y$ under the assumptions $\|y - y^\delta\| \le \delta$ and $x^\dagger \in M$? (ii) How to regularize such that the best possible accuracy can be guaranteed? We apply our results to Cauchy problems for the Helmholtz equation and show that, depending on different smoothness situations, the best possible accuracy may be of Hölder type, of logarithmic type or of some other type. In addition, we study regularization methods that provide the best possible accuracy. In case of appropriate *a posteriori* parameter choice, the best possible order of accuracy can be obtained without using any smoothness information for $x^\dagger$.

1. Introduction

In this paper we consider ill-posed problems

$$Ax = y \tag{1.1}$$

where $A : X \to Y$ is a bounded linear operator between infinite dimensional Hilbert spaces X and Y with non-closed range $\mathcal{R}(A)$. We shall denote the inner product and the corresponding norm on the Hilbert spaces by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. We assume throughout the paper that the operator A is injective and that the exact right hand side y belongs to $\mathcal{R}(A)$ so that (1.1) has a unique solution $x^\dagger \in X$. We are interested in problems (1.1) where instead of $y \in \mathcal{R}(A)$ we have a noisy right hand side $y^\delta \in Y$ with

$$\|y - y^\delta\| \le \delta. \tag{1.2}$$

Since $\mathcal{R}(A)$ is assumed to be non-closed, the solution $x^\dagger$ of problem (1.1) does not depend continuously on the data. Hence, the numerical treatment of problem (1.1), (1.2) requires the application of special regularization methods.

Linear ill-posed problems arise in different applications such as geophysics, finance, astronomy, biology, medicine, technology and others. Important examples

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are the identification of a derivative from noisy data (see, e. g., [7, 15]), deconvolution problems for modeling the problem of image deblurring (see, e. g., [4]), inverse heat conduction problems (see, e. g., [1, 3, 8]) or problems of computerized tomography (see, e. g., [22]). For some overview on further applications we recommend the books [5, 13].

In this paper we are interested in applications connected with Cauchy problems for the Helmholtz equation. Such problems arise, e.g., in optoelectronics, and in particular in laser beam models, see [2, 24, 25, 26]. For a mathematical formulation we follow the paper [25], denote by $r = (x, y)$ the first two variables and consider the Helmholtz equation

$$\left.\begin{aligned} \Delta u + k^2 u = 0 \qquad &\text{for} \quad (r, z) \in \Omega = \mathbb{R}^2 \times (0, d) \\ u(\cdot, z) \in L^2(\mathbb{R}^2) \qquad &\text{for} \quad z \in (0, d) \end{aligned}\right\} \tag{1.3}$$

where $\Delta u = u_{xx} + u_{yy} + u_{zz}$ and $k > 0$ is the wave number. Connected with (1.3) we define the space $H = L^2(\mathbb{R}^2)$ with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$ and formulate the following problem:

Problem P1 (Identification of $u(r, z)$ from $u(r, d)$)**.** Given $u_z(r, d) = 0$ and noisy data $u^\delta(r, d) \in H$ for $u(r, d)$ satisfying

$$\|u(\cdot, d) - u^\delta(\cdot, d)\| \le \delta,$$

find for some fixed $z \in [0, d)$ the solution $u(r, z)$ of problem (1.3).

The problem of identifying $u(r, z)$ from (unperturbed) data $u(r, d)$ can be formulated as an operator equation

$$A(z)u(r, z) = u(r, d), \quad A(z) \in \mathcal{L}(H, H) \tag{1.4}$$

which is a special case of the operator equation (1.1). However, the operator $A(z)$ is only bounded for small wave numbers $k(d - z) < \frac{\pi}{2}$, see Section 3. Hence, in the case of large wave numbers $k(d - z) \ge \frac{\pi}{2}$ the general results for operator equations (1.1) with bounded operators A cannot be applied.

The paper is organized as follows. In Section 2 we consider linear ill-posed problems (1.1) and discuss the question concerning the best possible accuracy for identifying $x^\dagger$ under the assumptions $\|y - y^\delta\| \le \delta$ and $x^\dagger \in M$ where M is some general source set. By using interpolation techniques we derive an explicit formula for the best possible worst case error and the best possible conditional stability estimate on a general source set M. In Section 3 we consider Problem P1 and discuss properties of the operator $A(z)$. We apply the general results of Section 2 and obtain best possible conditional stability estimates of Hölder type on the source set $M = M_E = \{u(\cdot, z) \in H \,|\, u \in D, \|u(\cdot, 0)\| \le E\}$, where D denotes the set of all solutions of (1.3). In Section 4 we study three generalizations for Problem P1. In a first generalization we allow more general source sets. In a second generalization we allow in (1.3) noisy data for $u_z(\cdot, d)$ with noise level δ and in a third generalization we allow noisy Dirichlet and Neumann data. Section 5 is devoted to regularization methods that provide optimal and order optimal error bounds on source sets M and in Section 6 we discuss the discrepancy principle for choosing the regularization parameter. This *a posteriori* rule adapts automatically to the generally unknown solution smoothness.

2. **Worst case analysis and conditional stability estimates**

2.1. **Worst case analysis.** Let us give some comment on order optimal convergence rates for identifying $x^\dagger$ from noisy data $y^\delta \in Y$ under the assumption (1.2) and $x \in M$ where $M \subset X$ is some centrally symmetric and convex set. We remember that a set M is called centrally symmetric if for all $x \in M$ also $-x$ is an element of M. We follow the books [11, 33] and start with some notational issues:

(i) Let $\mathcal{R} : Y \to X$ be an arbitrary operator and $\mathcal{R}(y^\delta)$ be an approximate solution for $x^\dagger$. Then the quantity
$$\Delta_{\mathcal{R}}(\delta, A, M) = \sup\left\{\|\mathcal{R}(y^\delta) - x\| \,\Big|\, \|Ax - y^\delta\| \le \delta,\ x \in M\right\}$$
is called *worst case error of the method $\mathcal{R}$ on the set M*. This quantity characterizes the accuracy of the method $\mathcal{R}$ in the worst case sense.

(ii) An optimal method $\mathcal{R}_{\mathrm{opt}}$ is characterized by
$$\Delta_{\mathcal{R}_{\mathrm{opt}}}(\delta, A, M) = \inf_{\mathcal{R}} \Delta_{\mathcal{R}}(\delta, A, M) \qquad (0 < \delta \le \delta_0)$$
and this quantity is called *best possible worst case error on the set M*.

(iii) The quantity
$$\omega(\delta, A, M) = \sup\left\{\|x\| \,\Big|\, x \in M,\ \|Ax\| \le \delta\right\} \tag{2.1}$$
is called *modulus of continuity of the inverse operator A^{-1} on the set M*.

The best possible worst case error obeys following estimate (see [11, 33]):

Proposition 2.1. *Let $M \subset X$ be centrally symmetric and convex. Then the best possible worst case error on M and the modulus of continuity of the inverse operator A^{-1} on M are related by*
$$\omega(\delta, A, M) \le \inf_{\mathcal{R}} \Delta_{\mathcal{R}}(\delta, A, M) \le 2\omega(\delta, A, M).$$

We are now interested in special sets
$$M = M_{\varphi,E} = \left\{x \in X \,\Big|\, x = [\varphi(A^*A)]^{1/2} v,\ \|v\| \le E\right\} \tag{2.2}$$
with some index function $\varphi(\lambda)$, $0 \le \lambda \le \|A^*A\|$. We note that

(i) according to [9, 17] a function $\varphi : (0, a] \to (0, b]$ is called an *index function* if it is continuous and monotonically increasing with $\lim\limits_{t \to +0} \varphi(t) = 0$,

(ii) any set defined by (2.2) is called *source set*.

The source sets defined by (2.2) are centrally symmetric, convex and bounded. The best possible worst case error and the modulus of continuity of the inverse operator A^{-1} on $M_{\varphi,E}$ obey following relations:

Proposition 2.2. *Let $M_{\varphi,E}$ be the source set (2.2) and let $\varrho(t) := t\varphi^{-1}(t)$.*

(i) *For the best possible worst case error on $M_{\varphi,E}$ there holds*
$$\inf_{\mathcal{R}} \Delta_{\mathcal{R}}(\delta, A, M_{\varphi,E}) = \omega(\delta, A, M_{\varphi,E}).$$

(ii) *If* $\delta^2/E^2 \in \sigma(G)$ *with* $G := A^*A\varphi(A^*A)$, *then*

$$\omega(\delta, A, M_{\varphi,E}) \geq E\sqrt{\varrho^{-1}(\delta^2/E^2)}.$$

(iii) *If the index function* ϱ *is convex, then*

$$\omega(\delta, A, M_{\varphi,E}) \leq E\sqrt{\varrho^{-1}(\delta^2/E^2)}.$$

The proof of part (i) of the proposition follows from a more general result in [10, Theorem 3]. For the proof of the parts (ii) and (iii) of the proposition see, e. g., [30, Theorem 2.1]. Note that the spectrum $\sigma(G)$ of the operator G obeys $\sigma(G) \subset [0, a\varphi(a)]$ with $a = \|A^*A\|$. Hence, in order to guarantee that $\delta^2/E^2 \in \sigma(G)$ we need that $\delta^2/E^2 \leq a\varphi(a)$. Due to Proposition 2.2, the magnitude $E\sqrt{\varrho^{-1}(\delta^2/E^2)}$ will serve us as benchmark for the best possible accuracy for identifying $x^\dagger$ from noisy data $y^\delta \in Y$ under the assumptions (1.2) and $x \in M_{\varphi,E}$.

2.2. **Conditional stability estimates.** The modulus of continuity of the inverse operator A^{-1} on the set M which is defined by (2.1) is closely related with the notion of conditional stability. An operator $A \in \mathcal{L}(X,Y)$ is called to satisfy a *conditional stability estimate on the set* M if there exists an index function $\beta : (0,a] \to (0,b]$ such that

$$\|x\| \leq \beta\left(\|Ax\|\right) \quad \text{for all } x \in M. \tag{2.3}$$

The function β is called *modulus of conditional stability of the operator* A^{-1} *on the set* M. As it can easily be seen, there always holds the estimate

$$\omega(\delta, A, M) \leq \beta(\delta). \tag{2.4}$$

Clearly, there are infinitely many moduli of conditional stability of A^{-1} on the special set $M_{\varphi,E}$ defined by (2.2). An *optimal* modulus of conditional stability $\beta_{\rm opt}$ satisfying (2.3) arises if there is equality in (2.4). Therefore, following definition makes sense.

Definition 2.3. We call $\beta_{\rm opt}$ as *best possible modulus of conditional stability of the operator* A^{-1} *on the set* $M_{\varphi,E}$ if $\beta_{\rm opt}$ satisfies (2.3) and if for all δ with $\delta^2/E^2 \in \sigma(A^*A\varphi(A^*A))$ there holds

$$\omega(\delta, A, M_{\varphi,E}) = \beta_{\rm opt}(\delta).$$

Interpolation techniques allow to derive a formula for the best possible modulus of conditional stability of the operator A^{-1} on the set $M_{\varphi,E}$.

Theorem 2.4. *Assume* $x \in M_{\varphi,E}$ *where* $M_{\varphi,E}$ *is given by (2.2). Assume further that* $\varrho(\lambda) := \lambda\varphi^{-1}(\lambda)$ *is convex. Then,*

$$\|x\| \leq \beta_{\rm opt}(\|Ax\|) \quad \textit{with } \beta_{\rm opt}(\delta) = E\sqrt{\varrho^{-1}(\delta^2/E^2)}. \tag{2.5}$$

Proof. We follow the ideas outlined in [30, Theorem 2.1]. Let E_λ the spectral family of A^*A. Since ϱ is convex we may employ Jensen's inequality and obtain due to $\varrho(\varphi(\lambda))[\varphi(\lambda)]^{-1} = \lambda$ that

$$\varrho\left(\frac{\|x\|^2}{\|[\varphi(A^*A)]^{-1/2}x\|^2}\right) \leq \frac{\int \varrho(\varphi(\lambda))[\varphi(\lambda)]^{-1}\,\mathrm{d}\|E_\lambda x\|^2}{\|[\varphi(A^*A)]^{-1/2}x\|^2} = \frac{\|Ax\|^2}{\|[\varphi(A^*A)]^{-1/2}x\|^2},$$

or equivalently,

$$\|[\varphi(A^*A)]^{-1/2}x\|^2\,\varrho\left(\frac{\|x\|^2}{\|[\varphi(A^*A)]^{-1/2}x\|^2}\right) \le \|Ax\|^2.$$

Since ϱ is convex, $t \to t^{-1}\varrho(t)$ is increasing. Consequently, $t \to t\varrho(1/t)$ is decreasing. Hence, since $\|[\varphi(A^*A)]^{-1/2}x\| \le E$, the above estimate gives

$$E^2\varrho(\|x\|^2/E^2) \le \|Ax\|^2.$$

Rearranging terms gives

$$\|x\| \le \beta\left(\|Ax\|\right) \quad \text{with} \quad \beta(\delta) = E\sqrt{\varrho^{-1}(\delta^2/E^2)}.$$

Due to part (ii) of Proposition 2.2 and estimate (2.4) we have for for all δ with $\delta^2/E^2 \in \sigma(A^*A\varphi(A^*A))$ the inequality chain $\beta(\delta) \le \omega(\delta, A, M_{\varphi,E}) \le \beta(\delta)$, which gives $\omega(\delta, A, M_{\varphi,E}) = \beta(\delta)$. That is, $\beta(\delta) = E\sqrt{\varrho^{-1}(\delta^2/E^2)}$ is the best possible modulus of conditional stability of the operator A^{-1} on the set $M_{\varphi,E}$. □

Our estimate (2.5) of Theorem 2.4 is a conditional stability estimate on the set $M_{\varphi,E}$ which cannot be improved. It tells us that for different data y_1, y_2 and corresponding solutions x_1, x_2 with $x_1 - x_2 \in M_{\varphi,E}$ we have the conditional stability estimate

$$\|x_1 - x_2\| \le E\sqrt{\varrho^{-1}(\|y_1 - y_2\|^2/E^2)}\,.$$

Let us give a further comment on our central Theorem 2.4. In many inverse partial differential equation problems this theorem makes it possible to derive explicit or implicit formulae for the best possible modulus of conditional stability on special sets M that arise by imposing a bound on a part of the solution of the partial differential equation. Special formulae that arise from (2.5) have successfully been derived for backward heat equation problems in one and more dimensions in [21, 27, 30], for sideways parabolic problems in [6, 29], for Cauchy problems for the Laplace equation in [28] for fractional differentiation problems in [31] and for Cauchy problems for the Helmholtz equation in [34]. In the next section we apply Theorem 2.4 for obtaining conditional stability estimates for special Cauchy problems for the Helmholtz equation. In particular, we extend the problems from [34] and improve the results. Our way of deriving formulae for conditional stability estimates consists roughly speaking in following three steps:

(1) Derive the index function φ such that the set M coincides with the set $M_{\varphi,E}$ given by (2.2).
(2) Compute the function $\varrho(\lambda) := \lambda\varphi^{-1}(\lambda)$ and prove its convexity.
(3) Derive a formula for β_{opt} given in (2.5).

Note that formulae for the modulus of conditional stability may sometimes also be obtained by so called logarithmic convexity arguments, see, e. g., [5, 12, 23].

3. **Cauchy problem for the Helmholtz equation**

3.1. **Operator equation formulation in the frequency space.** Transforming the operator equation (1.4) into the frequency domain provides (see [25, formula

(2.11)]) the equivalent operator equation

$$\widehat{A}(z)\widehat{u}(\xi,z)=\widehat{u}(\xi,d) \quad \Leftrightarrow \quad \frac{1}{\cosh\left((d-z)\sqrt{|\xi|^2-k^2}\right)}\widehat{u}(\xi,z)=\widehat{u}(\xi,d) \tag{3.1}$$

with $\xi=(\xi_1,\xi_2)$ and $|\xi|^2=\xi_1^2+\xi_2^2$ where $\widehat{A}(z)=\mathcal{F}A(z)\mathcal{F}^{-1}$, $\mathcal{F}\in\mathcal{L}(H,H)$ is the Fourier operator and $\widehat{u}(\xi,z)$ is the Fourier transform of $u(r,z)$ with respect to the variable $r=(x,y)$, that is,

$$\widehat{u}(\xi,z)=\mathcal{F}\left(u(r,z)\right)=\frac{1}{2\pi}\int_{\mathbb{R}^2}u(r,z)\mathrm{e}^{-\mathbf{i}\,r\cdot\xi}\,\mathrm{d}r.$$

3.2. **Properties and solution smoothness.** The operator $\widehat{A}(z)$ given in (3.1) is a multiplication operator. We use the identity

$$\cosh\Big((d-z)\sqrt{|\xi|^2-k^2}\Big)=\cos\Big((d-z)\sqrt{k^2-|\xi|^2}\Big) \quad \text{for} \quad |\xi|\le k$$

and realize that following properties are true:

(i) If there exists a solution of the equation (3.1), then this solution is unique.

(ii) The existence of a solution of equation (3.1) is guaranteed if the right hand side $\widehat{g}(\xi)=\widehat{u}(\xi,d)$ decays sufficiently fast for $|\xi|\to\infty$. More accurate, the existence of a solution is guaranteed if the Picard condition $\int_{\mathbb{R}^2}\cosh^2\left((d-z)\sqrt{|\xi|^2-k^2}\right)\widehat{g}^2(\xi)\,\mathrm{d}\xi<\infty$ is satisfied.

(iii) Since $\cosh\left((d-z)\sqrt{|\xi|^2-k^2}\right)\to\infty$ exponentially for $|\xi|\to\infty$, both inverse operators $A^{-1}(z)$ and $\widehat{A}^{-1}(z)$ are unbounded operators. It follows that both problems (1.4) and (3.1), respectively, are severely ill-posed problems. The ill-posedness becomes worse as z decreases.

(iv) Under the assumption $k(d-z)<\frac{\pi}{2}$ both operators $A(z)$ and $\widehat{A}(z)$ are linear bounded self-adjoint operators with spectrum in $[0,1/\cos k(d-z)]$. For $k(d-z)\ge\frac{\pi}{2}$ both operators $A(z)$ and $\widehat{A}(z)$ are unbounded.

In Figures 1 and 2 the multiplicator function

$$a(|\xi|)=1/\cosh\Big((d-z)\sqrt{|\xi|^2-k^2}\Big)$$

of the operator $\widehat{A}(z)$ of the operator equation (3.1) is displayed for $d=1$ and different values of k and z. Figure 1 illustrates the situation $k(d-z)<\frac{\pi}{2}$ in which the operator $\widehat{A}(z)$ is bounded by $1/\cos k(d-z)$.

Figure 2 illustrates the situation $k(d-z)>\pi/2$ in which the operator $\widehat{A}(z)$ is unbounded. Note that in the case $k(d-z)=\pi/2$ we have $\sigma(\widehat{A}(z))=[0,\infty)$, that in the case $\pi/2<k(d-z)<\pi$ we have $\sigma(\widehat{A}(z))=\mathbb{R}\setminus(\frac{1}{\cos k(d-z)},0)$ and that in the case $\pi\le k(d-z)<\infty$ we have $\sigma(\widehat{A}(z))=\mathbb{R}\setminus(-1,0)$.

Now let us impose some solution smoothness for the unknown solution $u(r,z)$ of the operator equation (1.4). Let $D\subset H^2(\Omega)$ denote the set of solutions of the Helmholtz equation (1.3). Clearly, since $u(r,z)$ satisfies the differential equation problem (1.3) we expect for any fixed $z\in[0,d)$ some inherent smoothness for

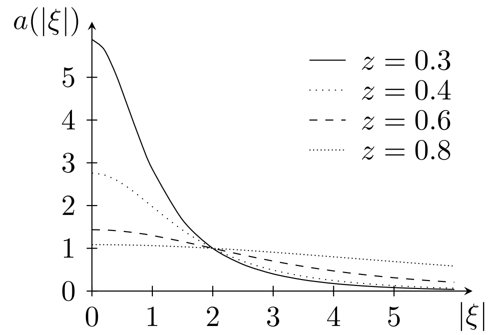

FIGURE 1. Multiplicator function $a(|\xi|)$ for $d = 1$, $k = 2$ and different z

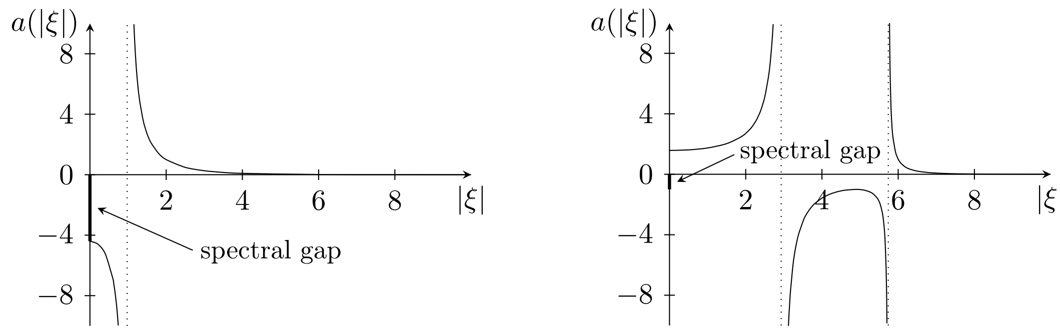

FIGURE 2. Multiplicator function $a(|\xi|)$ in the both cases $d = 1$, $k = 2$, $z = 0.1$ (left) and $d = 1$, $k = 6$, $z = 0.1$ (right)

$u(r, z)$ which may be expressed by

$$u(r, z) \in M = M_E = \Big\{u(\cdot, z) \in H \,\Big|\, u \in D,\ \|u(\cdot, 0)\| \le E\Big\} \tag{3.2}$$

with some $E > 0$. Now we ask the question if the set (3.2) is equivalent to some general source set

$$M_{\varphi,E} = \Big\{u(\cdot, z) \in H \,\Big|\, u(\cdot, z) = [\varphi(A^*(z)A(z))]^{1/2}v,\ \|v\| \le E\Big\} \tag{3.3}$$

with some index function $\varphi = \varphi(\lambda)$.

Proposition 3.1. *For Problem P1 we have equality $M_E = M_{\varphi,E}$ with M_E and $M_{\varphi,E}$ given by (3.2) and (3.3), respectively, if $\varphi(\lambda)$ is given (in parameter representation) by*

$$\left.\begin{aligned} \lambda(t) &= 1/\cosh^2\left((d - z)\sqrt{t^2 - k^2}\right) \\ \varphi(t) &= \cosh^2\left((d - z)\sqrt{t^2 - k^2}\right)/\cosh^2\left(d\sqrt{t^2 - k^2}\right) \end{aligned}\right\} \quad (0 \le t < \infty). \tag{3.4}$$

Proof. We observe that we have equality $M_E = M_{\varphi,E}$ if $\widehat{M}_E = \widehat{M}_{\varphi,E}$ where

$$\widehat{M}_E = \Big\{\widehat{u}(\cdot, z) \in H \,\Big|\, \widehat{u} \in \widehat{D},\ \|\widehat{u}(\cdot, 0)\| \le E\Big\},$$
$$\widehat{M}_{\varphi,E} = \Big\{\widehat{u}(\cdot, z) \in H \,\Big|\, \widehat{u}(\cdot, z) = [\varphi(\widehat{A}^*(z)\widehat{A}(z))]^{1/2}\widehat{v},\ \|\widehat{v}\| \le E\Big\}$$

and $\widehat{D} = \{\widehat{u}(\xi, z) \,|\, u \in D\}$. That is, we have equality $M_E = M_{\varphi,E}$ if $\|\widehat{u}(\cdot,0)\| = \|[\varphi(\widehat{A}^*(z)\widehat{A}(z))]^{-1/2}\widehat{u}(\cdot,z)\|$, or equivalently,

$$\varphi(\widehat{A}^*(z)\widehat{A}(z)) = \cosh^2\left((d-z)\sqrt{|\xi|^2-k^2}\right) / \cosh^2\left(d\sqrt{|\xi|^2-k^2}\right)$$

with

$$\widehat{A}^*(z)\widehat{A}(z) = 1/\cosh^2\left((d-z)\sqrt{|\xi|^2-k^2}\right).$$

That is, we have equality $M_E = M_{\varphi,E}$ for φ given by (3.4). $\square$

We observe following properties:

Case 1. If the wave number k satisfies $k < \frac{\pi}{2d}$, then the operator $\widehat{A}(z)$ is bounded and $\varphi = \varphi(\lambda)$ defined by (3.4) is an index function. It can be shown that the resulting index function $\varrho(\lambda) = \lambda\varphi^{-1}(\lambda)$ is convex for the whole t-range $t \in [0,\infty)$. The convexity proof for the restricted t-range $t \in [k,\infty)$ may be found in [28]. This Case 1 allows the application of Theorem 2.4 without any problems. Figure 3 illustrates the situation of this case.

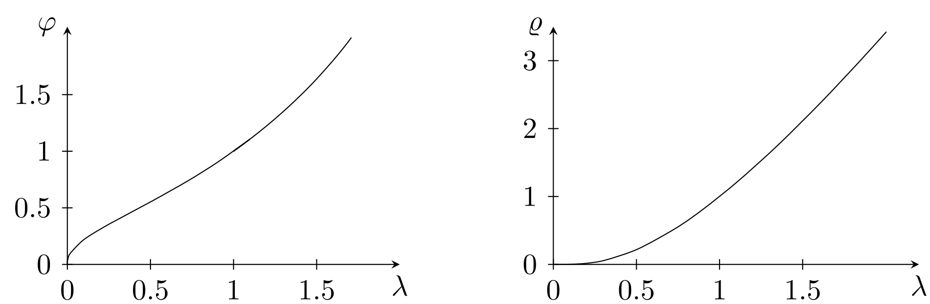

Figure 3. Source function φ for $d = 1$, $z = 0.3$, $k = 1$ (left) and corresponding function $\varrho(\lambda) = \lambda\varphi^{-1}(\lambda)$ (right)

Case 2. If the wave number k satisfies $\frac{\pi}{2d} \le k < \frac{\pi}{2(d-z)}$, then $\widehat{A}(z)$ is still bounded, however φ has poles and is therefore not an index function. If the wave number k is in the range $k \ge \frac{\pi}{2(d-z)}$, then the operator $\widehat{A}(z)$ is unbounded and φ has again poles and in addition different branches. As a consequence, Theorem 2.4 cannot be applied directly. We will handle this case by a special decomposition idea. Figure 4 illustrates the situation of Case 2 for $\frac{\pi}{2d} \le k < \frac{\pi}{2(d-z)}$.

3.3. **Conditional stability estimate for small wave numbers.** We study in this section the case of small wave numbers k satisfying $k < \frac{\pi}{2d}$.

Theorem 3.2. *Let the wave number k satisfy $k < \frac{\pi}{2d}$. Then the modulus of continuity and the best possible modulus of conditional stability of the inverse operator $\widehat{A}^{-1}(z)$ on the set $\widehat{M}_E := \left\{\widehat{u}(\cdot,z) \in H \,\middle|\, \widehat{u} \in \widehat{D}\,,\ \|\widehat{u}(\cdot,0)\| \le E\right\}$ are given by*

$$\omega(\delta, \widehat{A}(z), \widehat{M}_E) = \beta_{\mathrm{opt}}(\delta) = \delta\,\cosh\left(\frac{d-z}{d}\,\mathrm{arcosh}\frac{E}{\delta}\right). \tag{3.5}$$

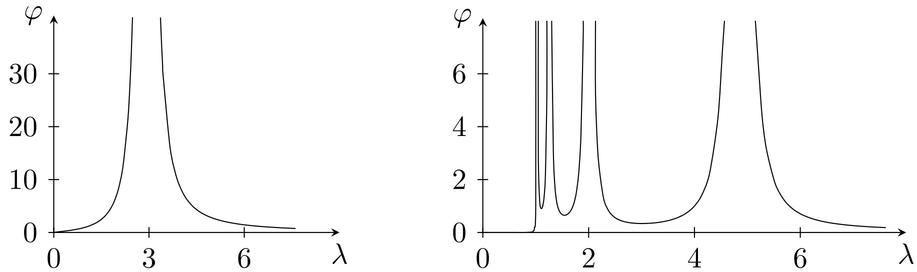

FIGURE 4. Source functions φ for $d = 1$, $z = 0.4$, $k = 2$ (left) and for $d = 1$, $z = 0.9$, $k = 12$ (right)

Proof. The set $\widehat{M}_E$ is equivalent to

$$\widehat{M}_{\varphi,E} = \left\{ \widehat{u}(\cdot, z) \in H \,\middle|\, \widehat{u}(\xi, z) = [\varphi(\widehat{A}^*(z)\widehat{A}(z))]^{1/2}\widehat{v}(\xi),\ \|\widehat{v}\| \le E \right\},$$

where $\varphi = \varphi(\lambda)$ is given by (3.4). The function φ is an index function and $\varrho(\lambda) := \lambda\varphi^{-1}(\lambda)$ is given (in parameter representation) by

$$\left.\begin{array}{l} \lambda(t) = \cosh^2\left((d-z)\sqrt{t^2-k^2}\right) / \cosh^2\left(d\sqrt{t^2-k^2}\right) \\ \varrho(t) = 1/\cosh^2\left(d\sqrt{t^2-k^2}\right) \end{array}\right\} \quad (0 \le t < \infty). \tag{3.6}$$

The function ϱ defined by (3.6) is convex (compare [28, Prop. 3.4] for the restricted range $t \in [k, \infty)$) and the inverse ϱ^{-1} is given by

$$\left.\begin{array}{r} \lambda(t) = 1/\cosh^2\left(d\sqrt{t^2-k^2}\right) \\ \varrho^{-1}(t) = \cosh^2\left((d-z)\sqrt{t^2-k^2}\right) / \cosh^2\left(d\sqrt{t^2-k^2}\right) \end{array}\right\} \quad (k \le t < \infty).$$

This function possesses the explicit form $\rho^{-1}(\lambda) = \lambda \cosh^2\left(\frac{d-z}{d}\operatorname{arcosh}\frac{1}{\sqrt{\lambda}}\right)$. Now we apply Theorem 2.4 and obtain (3.5). $\square$

Remark 3.3. We note that for $\delta \le E$ the best possible modulus of conditional stability β_{opt} from (3.5) can be estimated by

$$\beta_{\text{opt}}(\delta) \le E^{1-z/d}\delta^{z/d}. \tag{3.7}$$

That is, we have an upper bound which is of Hölder type. In the special case $k = 0$ this bound can also be obtained by logarithmic convexity arguments as outlined in [5, 12, 23]. For applying this concept to our Cauchy problem for the Helmholtz equation with $k = 0$ we introduce the differential operator L by $Lu = -u_{xx} - u_{yy}$ and execute following three steps:

(i) We define the function $F(z) := \|u(\cdot, z)\|^2$.

(ii) We show that $[\ln F(z)]'' = \dfrac{F''(z)F(z) - [F'(z)]^2}{F^2(z)} \ge 0.$

(iii) We conclude that $F(z) \le F^{1-z/d}(0)F^{z/d}(d)$ which gives (3.7).

For the proof of the right estimate of (ii) we observe that $F'(z) = 2(u, u_z)$ and $F''(z) = 2(u_z, u_z) + 2(u, u_{zz}) = 2(u_z, u_z) + 2(u, Lu)$. From the two identities

$$\frac{\mathrm{d}}{\mathrm{d}z}(u, Lu) = 2(u_z, Lu) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}z}(u_z, u_z) = 2(u_z, u_{zz}) = 2(u_z, Lu)$$

we conclude that $\frac{\mathrm{d}}{\mathrm{d}z}(u, Lu) = \frac{\mathrm{d}}{\mathrm{d}z}(u_z, u_z)$. Integration with respect to z and observing that $u_z(r, d) = 0$ yields $(u, Lu) = (u_z, u_z) + (u(r, d), Lu(r, d))$. Hence, $F''(z)$ attains the form $F''(z) = 4(u_z, u_z) + 2(u(r, d), Lu(r, d))$. That is, $[\ln F(z)]'' \geq 0$ is equivalent to

$$2|(u, u_z)|^2 \leq 2\|u_z\|^2\|u\|^2 + (u(r, d), Lu(r, d))\,\|u\|^2.$$

This estimate, however, is a consequence of the Cauchy-Schwarz inequality and the fact that the operator L is positive definite.

3.4. **Conditional stability estimate in the general case.** In this subsection we will show that the conditional stability estimate of Theorem 3.2 is also valid in the case of large wave numbers. However, in case of large wave numbers $k \geq \frac{\pi}{2d}$ our Theorem 2.1 cannot be applied directly for obtaining a formula for $\beta_{\mathrm{opt}}(\delta)$. Alternatively, we can use a decomposition idea as outlined in [34] that makes it possible to apply Theorem 2.4. Our decomposition consists in

(i) decomposing $\mathbb{R}^2$ into the ill-posed part I and the well-posed part W where

$$I = \Big\{\xi \in \mathbb{R}^2 \,\Big|\, |\xi| \geq k\Big\} \quad \text{and} \quad W = \Big\{\xi \in \mathbb{R}^2 \,\Big|\, |\xi| \leq k\Big\},$$

(ii) decomposing the space $H = L^2(\mathbb{R}^2)$ into the direct sum

$$H = H_1 \oplus H_2 \quad \text{with} \quad H_1 = L^2(I) \quad \text{and} \quad H_2 = L^2(W),$$

(iii) decomposing the elements $\widehat{u}(\xi, z)\ \forall z \in [0, d]$ into the sum

$$\widehat{u}(\xi, z) = \widehat{u}_1(\xi, z) + \widehat{u}_2(\xi, z)$$

where $\widehat{u}_1(\cdot, z) := P_1\widehat{u}(\cdot, z)$, $\widehat{u}_2(\cdot, z) := P_2\widehat{u}(\cdot, z)$ and P_1, P_2 are the orthoprojections onto H_1 and H_2, respectively,

(iv) decomposing the set of solutions $\widehat{D}$ into the direct sum

$$\widehat{D} = \widehat{D}_1 \oplus \widehat{D}_2 \quad \text{with} \quad \widehat{D}_1 = P_1\widehat{D} \quad \text{and} \quad \widehat{D}_2 = P_2\widehat{D},$$

(v) decomposing the operator equation (3.1) in the frequency domain into two separate problems, one ill-posed problem

$$\widehat{A}_1(z)\widehat{u}_1(\xi, z) = \widehat{u}_1(\xi, d), \quad \widehat{A}_1 \in \mathcal{L}(H_1, H_1)$$

and one well-posed problem

$$\widehat{A}_2(z)\widehat{u}_2(\xi, z) = \widehat{u}_2(\xi, d), \quad \widehat{A}_2 \in \mathcal{L}(H_2, H_2),$$

(vi) decomposing the set $\widehat{M}_E = \Big\{\widehat{u}(\cdot, z) \in H \,\Big|\, \widehat{u} \in \widehat{D},\ \|\widehat{u}(\cdot, 0)\| \leq E\Big\}$ into two subsets

$$\widehat{M}_{1,E} = \Big\{\widehat{u}(\cdot, z) \in H_1 \,\Big|\, \widehat{u} \in \widehat{D}_1,\ \|\widehat{u}(\cdot, 0)\|_{H_1} \leq E\Big\},$$

$$\widehat{M}_{2,E} = \Big\{\widehat{u}(\cdot, z) \in H_2 \,\Big|\, \widehat{u} \in \widehat{D}_2,\ \|\widehat{u}(\cdot, 0)\|_{H_2} \leq E\Big\}.$$

Let $\varphi_1(\lambda)$ be defined by parameter representation (similarly as $\varphi(\lambda)$ in (3.4)), but now the range of the parameter t is restricted to $[k,\infty)$:

$$\left.\begin{aligned} \lambda(t) &= 1/\cosh^2\left((d-z)\sqrt{t^2-k^2}\right) \\ \varphi_1(t) &= \cosh^2\left((d-z)\sqrt{t^2-k^2}\right)/\cosh^2\left(d\sqrt{t^2-k^2}\right) \end{aligned}\right\} \quad (k \le t < \infty). \quad (3.8)$$

Then, in analogy to Proposition 3.1 we have

Proposition 3.4. *If $M_{\varphi_1,E}$ is given by (3.3) then we have equality $M_{1,E} = M_{\varphi_1,E}$.*

Since the function φ_1 is an index function, for the best possible modulus of conditional stability of the operator $\widehat{A}_1^{-1}(z)$ on the set $\widehat{M}_{1,E}$ we have in analogy to Theorem 3.2 the following result:

Proposition 3.5. *The modulus of continuity and the best possible modulus of conditional stability of the operator $\widehat{A}_1^{-1}(z)$ on the set $\widehat{M}_{1,E}$ are given by*

$$\omega(\delta, \widehat{A}_1(z), \widehat{M}_{1,E}) = \beta_{\rm opt}(\delta) = \delta \cosh\left(\frac{d-z}{d}\operatorname{arcosh}\frac{E}{\delta}\right).$$

For the well-posed part we have $\|\widehat{A}_2^{-1}(z)\| \le 1$. That is, for all elements $\widehat{u} \in H_2 = L^2(W)$ we have the estimate $\|\widehat{u}\| \le \|\widehat{A}_2(z)\widehat{u}\|$. Hence, the modulus of continuity and the best possible modulus of conditional stability of $\widehat{A}_2^{-1}(z)$ on an arbitrary subset of H_2 can be estimated by δ and we have

$$\omega(\delta, \widehat{A}_2(z), \widehat{M}_{2,E}) \le \delta.$$

From this property, Proposition 3.5 and the Pythagoras Theorem we get

Theorem 3.6. *The modulus of continuity and the best possible modulus of conditional stability of the inverse operator $\widehat{A}^{-1}(z)$ on the set*

$$\widehat{M}_E = \left\{\widehat{u}(\cdot,z) \in H \,\middle|\, \widehat{u} \in \widehat{D},\ \|\widehat{u}(\cdot,0)\| \le E\right\}$$

are given by the formula (3.5) for arbitrary wave number k.

This in turn implies that the modulus of continuity of the inverse operator $A^{-1}(z)$ and the best possible modulus of conditional stability of the operator $A^{-1}(z)$ on the set (3.2) are also given by the right hand side of (3.5). For $\delta \le E$ the right hand side of (3.5) can be estimated by $E^{1-z/d}\delta^{z/d}$ which is an upper bound of Hölder type.

4. Generalizations

In this section we discuss three generalizations of our model problem P1 from Section 1. In a first subsection we allow more general source sets. In a second subsection we treat the case of noisy Neumann data at $z = d$ and in a third subsection we discuss the general case of noisy Dirichlet and Neumann data. This general case is illustrated in Figure 5.

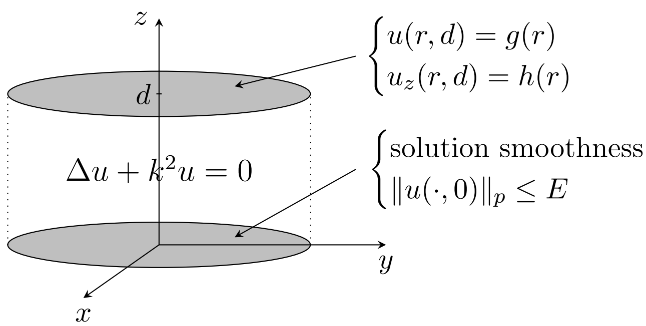

FIGURE 5. Extension of Problem P1: Identify $u(r,z) \in H$ for fixed $z \in [0,d)$ from noisy data $g^\delta(r) \in H$ and $h^\delta(r) \in H$ with $\|g - g^\delta\| \le \delta$ and $\|h - h^\delta\| \le \delta$

4.1. **More general source sets.** In this subsection we reconsider Problem P1 and allow more general source sets, that is, instead of (3.2) we allow more general solution smoothness

$$u(r,z) \in M = M_{p,E} = \left\{ u(\cdot,z) \in H \,\middle|\, u \in D,\ \|u(\cdot,0)\|_p \le E \right\} \tag{4.1}$$

with some generally unknown $p \ge 0$. In (4.1) the norm $\|\cdot\|_p$ is the norm in the classical Sobolev scale $(H_p)_{p\in\mathbb{R}}$ (see [14]), that is,

$$\|w\|_p = \left(\int_{\mathbb{R}^2} \left(1 + |\xi|^2\right)^p |\widehat{w}|^2 \mathrm{d}\xi \right)^{1/2}.$$

Note that for larger p the smoothness assumption (4.1) becomes more restrictive. In analogy to Proposition 3.1 we have

Proposition 4.1. *For Problem P1 we have equality $M_{p,E} = M_{\varphi,E}$ with $M_{p,E}$ and $M_{\varphi,E}$ given by (4.1) and (3.3), respectively, if $\varphi(\lambda)$ is given (in parameter representation) by*

$$\left.\begin{aligned} \lambda(t) &= \frac{1}{\cosh^2\left((d-z)\sqrt{t^2-k^2}\right)} \\ \varphi(t) &= \frac{\cosh^2\left((d-z)\sqrt{t^2-k^2}\right)}{(1+t^2)^p \cosh^2\left(d\sqrt{t^2-k^2}\right)} \end{aligned}\right\} \quad (0 \le t < \infty). \tag{4.2}$$

The function φ implicitly defined by (4.2) is an index function for small wave numbers k satisfy $k < \frac{\pi}{2d}$. From (4.2) we have that in this case the function $\varrho(\lambda) := \lambda\varphi^{-1}(\lambda)$ is given (in parameter representation) by

$$\left.\begin{aligned} \lambda(t) &= \frac{\cosh^2\left((d-z)\sqrt{t^2-k^2}\right)}{(1+t^2)^p \cosh^2\left(d\sqrt{t^2-k^2}\right)} \\ \varrho(t) &= \frac{1}{(1+t^2)^p \cosh^2\left(d\sqrt{t^2-k^2}\right)} \end{aligned}\right\} \quad (0 \le t < \infty).$$

Theorem 4.2. *The modulus of continuity and the best possible modulus of conditional stability of the inverse operator $A^{-1}(z)$ on the set $M_{p,E}$ are given by*

$$\omega(\delta, A(z), M_{p,E}) = \beta_{\text{opt}}(\delta) = E\,\frac{\cosh\left((d-z)\sqrt{t_0^2-k^2}\right)}{(1+t_0^2)^{p/2}\cosh\left(d\sqrt{t_0^2-k^2}\right)} \tag{4.3}$$

where t_0 is the solution of the equation

$$\frac{1}{(1+t^2)^{p/2}\cosh\left(d\sqrt{t^2-k^2}\right)} = \frac{\delta}{E}.$$

For $\delta \to 0$ we have the asymptotic representation

$$\omega(\delta, A(z), M_{p,E}) = \beta_{\text{opt}}(\delta) = E^{1-z/d}\left(\frac{\delta}{2}\right)^{z/d}\left[\frac{1}{d}\ln\frac{E}{\delta}\right]^{-p(1-z/d)}(1+o(1))\,. \tag{4.4}$$

Proof. For the wave number k satisfying $k < \frac{\pi}{2d}$ we can apply our general Theorem 2.4 and obtain (4.3). By using the decomposition idea of Subsection 3.4 it can be shown that this result is also valid in case of large wave numbers $k \geq \frac{\pi}{2d}$. □

Remark 4.3. From (4.4) we conclude that for $p > 0$ the modulus of continuity and the best possible modulus of conditional stability improve slightly by the logarithmic factor $\left[\frac{1}{d}\ln\frac{E}{\delta}\right]^{-p(1-z/d)}$. In addition, formula (4.4) shows that for $z = 0$ and $p > 0$ we have logarithmic stability.

4.2. **Noisy Neumann data.** Connected with equation (1.3) we consider

Problem P2 (Identification of $u(r,z)$ from $u_z(r,d)$)**.** Given $u(r,d) = 0$ and noisy data $u_z^\delta(r,d) \in H$ for $u_z(r,d)$ satisfying

$$\|u_z(\cdot,d) - u_z^\delta(\cdot,d)\| \leq \delta,$$

find for some fixed $z \in [0,d)$ the solution $u(r,z)$ of problem (1.3).

The problem of identifying $u(r,z)$ from (unperturbed) data $u_z(r,d)$ can be formulated as an operator equation

$$A(z)u(r,z) = u_z(r,d), \quad A(z) \in \mathcal{L}(H,H) \tag{4.5}$$

which is a special case of the operator equation (1.1). Transforming the operator equation (4.5) into the frequency domain provides the equivalent operator equation

$$\widehat{A}(z)\widehat{u}(\xi,z) = \widehat{u}_z(\xi,d) \quad \Leftrightarrow \quad \frac{-\sqrt{|\xi|^2-k^2}}{\sinh\left((d-z)\sqrt{|\xi|^2-k^2}\right)}\widehat{u}(\xi,z) = \widehat{u}_z(\xi,d). \tag{4.6}$$

The operator $\widehat{A}(z)$ given in (4.6) is a multiplication operator with the multiplicator function $a(|\xi|) = -\sqrt{|\xi|^2-k^2}/\sinh\left((d-z)\sqrt{|\xi|^2-k^2}\right)$. We use the identity

$$\sinh\Big((d-z)\sqrt{|\xi|^2-k^2}\Big) = \mathrm{i}\,\sin\Big((d-z)\sqrt{k^2-|\xi|^2}\Big) \quad \text{for} \quad |\xi| \leq k$$

and realize that following properties are true:

(i) If there exists a solution of the equation (4.5), then this solution is unique.

(ii) The existence of a solution of equation (4.5) is guaranteed if the right hand side $\widehat{h}(\xi) = \widehat{u}_z(\xi, d)$ decays sufficiently fast for $|\xi| \to \infty$. More accurate, the existence of a solution is guaranteed if the Picard condition $\int_{\mathbb{R}^2} \left(|\xi|^2 - k^2\right)^{-1} \sinh^2\left((d-z)\sqrt{|\xi|^2 - k^2}\right) \widehat{h}^2(\xi)\, \mathrm{d}\xi < \infty$ is satisfied.

(iii) Since $(|\xi|^2 - k^2)^{-1} \sinh^2\left((d-z)\sqrt{|\xi|^2 - k^2}\right) \to \infty$ for $|\xi| \to \infty$, both inverse operators $A^{-1}(z)$ and $\widehat{A}^{-1}(z)$ are unbounded operators. It follows that both problems (4.5) and (4.6), respectively, are ill-posed problems. The ill-posedness becomes worse as z decreases.

(iv) Under the assumption $k(d-z) < \pi$ both operators $A(z)$ and $\widehat{A}(z)$ are linear bounded self-adjoint operators with spectrum in $[-k/\sin k(d-z), 0]$. For $k(d-z) \geq \pi$ both operators $A(z)$ and $\widehat{A}(z)$ are unbounded.

(iv) Note that in the case $k(d-z) = \pi$ we have $\sigma(\widehat{A}(z)) = (-\infty, 0]$, that in the case $\pi < k(d-z) < \frac{3\pi}{2}$ we have $\sigma(\widehat{A}(z)) = \mathbb{R} \setminus (0, \frac{-k}{\sin k(d-z)})$ and that in the case $\frac{3\pi}{2} \leq k(d-z) < \infty$ we have $\sigma(\widehat{A}(z)) = \mathbb{R} \setminus (0, k)$.

In our next proposition we look for a function φ such that $M_{p,E} = M_{\varphi,E}$ holds. We proceed as in the proof of Proposition 3.1 and obtain

Proposition 4.4. *For Problem P2 we have equality $M_{p,E} = M_{\varphi,E}$ with $M_{p,E}$ and $M_{\varphi,E}$ given by (4.1) and (3.3), respectively, if $\varphi(\lambda)$ is given (in parameter representation) by*

$$\left.\begin{aligned} \lambda(t) &= \frac{t^2 - k^2}{\sinh^2\left((d-z)\sqrt{t^2-k^2}\right)} \\ \varphi(t) &= \frac{\sinh^2\left((d-z)\sqrt{t^2-k^2}\right)}{(1+t^2)^p \sinh^2\left(d\sqrt{t^2-k^2}\right)} \end{aligned}\right\} \quad (0 \leq t < \infty). \tag{4.7}$$

The function φ implicitly defined by (4.7) is an index function for small wave numbers k satisfy $k < \frac{\pi}{d}$. From (4.7) we have that in this case the function $\varrho(\lambda) := \lambda\varphi^{-1}(\lambda)$ is given (in parameter representation) by

$$\left.\begin{aligned} \lambda(t) &= \frac{\sinh^2\left((d-z)\sqrt{t^2-k^2}\right)}{(1+t^2)^p \sinh^2\left(d\sqrt{t^2-k^2}\right)} \\ \varrho(t) &= \frac{t^2 - k^2}{(1+t^2)^p \sinh^2\left(d\sqrt{t^2-k^2}\right)} \end{aligned}\right\} \quad (0 \leq t < \infty).$$

We use the decomposition idea of Subsection 3.4, apply our general Theorem 2.4 and obtain

Theorem 4.5. *For Problem P2 the modulus of continuity and the best possible modulus of conditional stability of the inverse operator $A^{-1}(z)$ on the set $M_{p,E}$ are given by*

$$\omega(\delta, A(z), M_{p,E}) = \beta_{\mathrm{opt}}(\delta) = E\, \frac{\sinh\left((d-z)\sqrt{t_0^2 - k^2}\right)}{(1+t_0^2)^{p/2} \sinh\left(d\sqrt{t_0^2-k^2}\right)} \tag{4.8}$$

where t_0 is the solution of the equation

$$\frac{\sqrt{t^2-k^2}}{(1+t^2)^{p/2}\sinh\left(d\sqrt{t^2-k^2}\right)}=\frac{\delta}{E}.$$

For $\delta\to 0$ we have the asymptotic representation

$$\omega(\delta, A(z), M_{p,E}) = E^{1-z/d}\left(\frac{\delta}{2}\right)^{z/d}\left[\frac{1}{d}\ln\frac{E}{\delta}\right]^{-p(1-z/d)-z/d}(1+o(1)). \tag{4.9}$$

Proof. For the small wave numbers $k<\frac{\pi}{d}$ we apply Theorem 2.4. By using the decomposition idea of Subsection 3.4 it can be shown that the result of Theorem 4.5 is also valid in case of large wave numbers $k\geq\frac{\pi}{d}$. ☐

Remark 4.6. From (4.9) we conclude that for $p>0$ the modulus of continuity improves slightly by the logarithmic factor $\left[\frac{1}{d}\ln\frac{E}{\delta}\right]^{-p(1-z/d)-z/d}$. In addition, formula (4.9) shows that for $z=0$ and $p>0$ we have logarithmic stability. Comparing the asymptotic representations (4.4) for Problem P1 and (4.9) for Problem P2, respectively, we realize that for $z>0$ the modulus of continuity for Problem P2 is better by the logarithmic factor $\left[\frac{1}{d}\ln\frac{E}{\delta}\right]^{-z/d}$.

4.3. **Noisy Dirichlet and Neumann data.**

Connected with equation (1.3) we consider

Problem P3 (Identification of $u(r,z)$ from $g(r):=u(r,d)$ and $h(r):=u_z(r,d)$)**.** Given noisy data $u^\delta(r,d)\in H$ for $u(r,d)$ and $u_z^\delta(r,d)\in H$ for $u_z(r,d)$ satisfying

$$\|u(\cdot,d)-u^\delta(\cdot,d)\|\leq\delta_1 \quad\text{and}\quad \|u_z(\cdot,d)-u_z^\delta(\cdot,d)\|\leq\delta_2,$$

find for some fixed $z\in[0,d)$ the solution $u(r,z)$ of problem (1.3).

The problem of identifying $u(r,z)$ from (unperturbed) data $u(r,d)$ and $u_z(r,d)$ can be decomposed into two separate problems: P1 with zero Neumann data and P2 with zero Dirichlet data. The corresponding solutions will be denoted by $u_1(r,z)$ and $u_2(r,z)$, respectively.

Remark 4.7. If the exact data are such that solutions $u_1(r,z)$ and $u_2(r,z)$ exist, then there exists a solution $u(r,z)$ of Problem P3 which has the form $u(r,z)=u_1(r,z)+u_2(r,z)$, or equivalently, $\widehat{u}(\cdot,z)=\widehat{A}^{-1}(z)\widehat{g}(\cdot)+\widehat{B}^{-1}(z)\widehat{h}(\cdot)$, where $\widehat{A}(z)$ is the forward map of (3.1) and $\widehat{B}(z)$ is the forward map of (4.6). Moreover, by triangle inequality and the results of Theorems 4.2 and 4.5 we obtain for Problem P3 that under the a priori assumption $\|u(\cdot,0)\|_p\leq E$ we have for all $z\in[0,d]$ the stability estimate

$$\|u(\cdot,z)\|\leq\beta^A_{\text{opt}}(\|g\|)+\beta^B_{\text{opt}}(\|h\|)$$

with β^A_{opt} defined by (4.3) and β^B_{opt} defined by (4.8).

Assume now that for the exact data the Problem P3 has a solution. We can decompose it into the sum of a solution $v(r,z)$ of the boundary value problem (1.3) with boundary conditions $v(r,0)=0$ and $v_z(r,d)=h(r)$, which will be denoted by Problem P4, and the solution $w(r,z)$ of Problem P1 with boundary data $w(r,d) = g(r) - v(r,d)$. It can be proved (see [25, Lemma 2.1]) that for small wave numbers $k < \frac{\pi}{2d}$ the Problem P4 is well posed. Thus, Problem P1 with data $g(r) - v(r,d)$ has a solution and $u(r,z) = v(r,z) + w(r,z)$. Now, following the proof of [25, Lemma 2.1] and Theorem 4.2, we obtain the following result:

Theorem 4.8. *Let the wave number k satisfy $k < \frac{\pi}{2d}$. Then a solution of Problem P3 is the sum of a solution $v(\cdot, z)$ of a well posed boundary value problem with inverse operator bounded by*

$$C = \sqrt{d} \max \left\{ d, \frac{1}{k} \tan(dk) \right\}$$

and the solution $w(\cdot, z)$ of Problem P1 with data $g(\cdot) - v(\cdot, d)$. Under the a priori assumption $\|w(\cdot, 0)\|_p \le E$, the solution $w(\cdot, z)$ obeys for all $z \in [0, d]$ the stability estimate

$$\|w(\cdot, z)\| \le \beta_{\rm opt}(\|g(\cdot) - v(\cdot, d)\|)$$

with $\beta_{\rm opt}$ defined by (4.3).

5. Regularization

In the foregoing sections we have clarified the question which best possible error bound can be obtained for identifying $x^\dagger \in X$ from noisy data $y^\delta \in Y$ under the assumptions $\|y - y^\delta\| \le \delta$ and $x^\dagger \in M_{\varphi,E}$ given by (2.2). Now we will look for special regularization methods that guarantee this accuracy. We will distinguish our studies into regularization in case of known solution smoothness and regularization in case of unknown solution smoothness. We start by introducing a general regularization scheme.

5.1. A general regularization scheme.

Let us consider a general regularization scheme in which the regularized solutions with exact and noisy data y and y^δ, respectively, are defined by

$$x_\alpha = G g_\alpha(T^*T) T^* y \,, \quad x_\alpha^\delta = G g_\alpha(T^*T) T^* y^\delta \quad \text{with} \quad T = AG. \tag{5.1}$$

Here $G : X \to X$ is some linear self-adjoint operator that controls the smoothness to be introduced into the regularization and $g_\alpha : (0, \|T\|^2] \to \mathbb{R}$ is a piecewise continuous nonnegative function with the property that $\lim\limits_{\alpha \to 0+} g_\alpha(\lambda) = 1/\lambda$. Different regularization methods are characterized by different operators G and functions g_α in (5.1). Let us discuss some methods that fit into the framework of the general regularization scheme (5.1).

Example 5.1 (Ordinary Tikhonov regularization)**.** This method is characterized by (5.1) with $g_\alpha(\lambda) = 1/(\lambda + \alpha)$. The regularized solution x_α^δ can be obtained by solving the minimization problem

$$\min_{x \in X} J_\alpha(x) \,, \quad J_\alpha(x) = \|Ax - y^\delta\|^2 + \alpha \|G^{-1} x\|^2,$$

that is, by solving the Euler equation $(A^*A + \alpha G^{-2}) x_\alpha^\delta = A^* y^\delta$.

Example 5.2 (Tikhonov regularization of order m)**.** These methods are characterized by (5.1) with $g_\alpha(\lambda) = \left(1 - \left(\frac{\alpha}{\lambda+\alpha}\right)^m\right)/\lambda$. The regularized solutions $x_\alpha^\delta := x_{\alpha,m}^\delta$ can be obtained by solving the m operator equations

$$(A^*A + \alpha G^{-2})x_{\alpha,k}^\delta = A^*y^\delta + \alpha G^{-2}x_{\alpha,k-1}^\delta\,, \quad k = 1,\ldots,m\,, \quad x_0^\delta = 0.$$

For $m = 1$, this method coincides with the method of Example 5.1.

Example 5.3 (Asymptotical regularization)**.** This method is characterized by (5.1) with $g_\alpha(\lambda) = (1 - e^{-\lambda/\alpha})/\lambda$. In this method one solves the Cauchy problem

$$G^{-2}\dot{u}(t) + A^*Au(t) = A^*y^\delta\,, \quad 0 < t \le \tau\,, \quad u(0) = 0$$

and the regularized solution is defined by $x_\alpha^\delta = u(\tau)$. Here τ and α are related by $\tau = 1/\alpha$. For $G = I$ this method is known as *Showalter's method.*

Example 5.4 (Explicit iteration scheme)**.** As a special case of more general explicit iteration methods, let us consider the *Landweber iteration.* This method is characterized by (5.1) with $g_\alpha(\lambda) = \left(1 - (1-\mu\lambda)^{1/\alpha}\right)/\lambda$ with some constant $\mu \in (0, 1/\|G^2A^*A\|]$. The regularized solution $x_\alpha^\delta := u_n^\delta$ can be obtained by performing n iterations according to

$$u_k^\delta = u_{k-1}^\delta - G^2A^*(Au_{k-1}^\delta - y^\delta)\,, \quad k = 1,\ldots,n$$

with $u_0^\delta = 0$. Here, n and α are related by $\alpha = 1/n$.

Example 5.5 (Implicit iteration scheme)**.** This method is characterized by (5.1) with $g_\alpha(\lambda) = \left(1 - \left(\frac{\mu}{\lambda+\mu}\right)^{1/\alpha}\right)/\lambda$. The regularized solution $x_\alpha^\delta := u_n^\delta$ can be obtained by by solving the n operator equations

$$(A^*A + \mu G^{-2})u_k^\delta = A^*y^\delta + \mu G^{-2}u_{k-1}^\delta\,, \quad k = 1,\ldots,n$$

with $u_0^\delta = 0$. Here, n and α are related by $\alpha = 1/n$. For $\mu = 1$ and $G = I$ we have *Lardy's method.*

Example 5.6 (Spectral method)**.** This method (also called *spectral cutt-off*) is characterized by (5.1) with

$$g_\alpha(\lambda) = \begin{cases} 1/\lambda & \text{for } \lambda \ge \alpha \\ 0 & \text{for } \lambda < \alpha. \end{cases}$$

For problems with compact operators A and G, the numerical computation of x_α^δ can be done by

$$x_\alpha^\delta = \sum_{s_i \ge \sqrt{\alpha}} \frac{(y^\delta, v_i)}{s_i}\, u_i$$

where $\{s_i, u_i, v_i\}_{i\in\mathbb{N}}$ is the generalized singular system of the operator A satisfying $A^*Au_i = \lambda_i G^{-2}u_i$, $s_i = \sqrt{\lambda_i}$ and $v_i = \frac{1}{s_i}Au_i$. In fact, $\{s_i, u_i, v_i\}_{i\in\mathbb{N}}$ is a singular system of the compact operator $T = AG$.

Example 5.7 (Modified spectral method)**.** This method is characterized by

$$g_\alpha(\lambda) = \begin{cases} 1/\lambda & \text{for } \lambda \geq \alpha \\ 1/\sqrt{\alpha\lambda} & \text{for } \lambda < \alpha. \end{cases}$$

Let $T = AG$ be compact with singular system $\{s_i, u_i, v_i\}_{i\in\mathbb{N}}$. In this case the regularized solution (5.1) is given by

$$x_\alpha^\delta = \sum_{s_i \geq \sqrt{\alpha}} \frac{(y^\delta, v_i)}{s_i} u_i + \sum_{s_i < \sqrt{\alpha}} \frac{(y^\delta, v_i)}{\sqrt{\alpha}} u_i.$$

If one wants to apply a special regularization method, one has to make different decisions: First, one has to choose the function g_α, second one has to choose the operator G and third one has to choose the regularization parameter α. For a wrong choice of g_α, G or α, one may get bad regularized solutions x_α^δ. Results on error bounds for $\|x_\alpha^\delta - x^\dagger\|$ may be helpful for a proper choice of g_α, G and α. For deriving order optimal error bounds for $\|x_\alpha^\delta - x^\dagger\|$ with x_α^δ defined by (5.1) the following error representations with $T = AG$ will be useful:

$$\begin{aligned} x_\alpha^\delta - x_\alpha &= G g_\alpha(T^*T)T^*(y^\delta - y), \\ x^\dagger - x_\alpha &= G[I - g_\alpha(T^*T)T^*T]G^{-1}x^\dagger. \end{aligned}$$

For an analysis of these two error parts under different assumptions we recommend the papers [18, 19] and the references cited there.

Constructing regularized solutions for our problem (1.4) with noisy data $u^\delta(r, d)$ by the regularization methods of Examples 5.6 and 5.7 gives

$$u_\alpha^\delta(r, z) = \mathcal{F}^{-1}(\widehat{u}_\alpha^\delta(\xi, z)) \tag{5.2}$$

where the regularized solutions in the frequency domain are given as follows:

(i) For the spectral method of Example 5.6 we have

$$\widehat{u}_\alpha^\delta(\xi, z) = \begin{cases} \left(1/\sqrt{\lambda(|\xi|)}\right) \widehat{u}^\delta(\xi, d) & \text{for} \quad \lambda(|\xi|) \geq \alpha \\ 0 & \text{for} \quad \lambda(|\xi|) < \alpha \end{cases} \tag{5.3}$$

with $\lambda(|\xi|) = 1/\cosh^2\left((d - z)\sqrt{|\xi|^2 - k^2}\right)$. Note that for $k(d - z) < \pi/2$ the function $|\xi| \to \lambda(|\xi|)$ is monotonically decreasing for $|\xi| \in (0, \infty)$. Moreover, for arbitrary k, the function $\lambda(|\xi|)$ is monotonically decreasing on the domain $[k, \infty)$.

(ii) For the modified spectral method of Example 5.7 we have

$$\widehat{u}_\alpha^\delta(\xi, z) = \begin{cases} \left(1/\sqrt{\lambda(|\xi|)}\right) \widehat{u}^\delta(\xi, d) & \text{for} \quad \lambda(|\xi|) \geq \alpha \\ (1/\sqrt{\alpha}\,)\ \widehat{u}^\delta(\xi, d) & \text{for} \quad \lambda(|\xi|) < \alpha. \end{cases} \tag{5.4}$$

We note that for computational purposes we have the following equivalent representation $u_\alpha^\delta(r, z) = (1/\sqrt{\alpha}\,)\ u^\delta(r, d) + \mathcal{F}^{-1}(\widehat{w}_\alpha^\delta(\xi, z))$ with

$$\widehat{w}_\alpha^\delta(\xi, z) = \begin{cases} \left(1/\sqrt{\lambda(|\xi|)} - 1/\sqrt{\alpha}\right) \widehat{u}^\delta(\xi, d) & \text{for} \quad \lambda(|\xi|) \geq \alpha \\ 0 & \text{for} \quad \lambda(|\xi|) < \alpha. \end{cases} \tag{5.5}$$

As we can see, in contrast to the regularization methods discussed in the above Examples 5.1 – 5.5, the computation of both regularized solutions (5.3) and (5.5) does not require the spectrum $\lambda(|\xi|)$ of the operator $\widehat{A}(z)$ for $\lambda(|\xi|) < \alpha$.

5.2. **Optimal regularization.** In this subsection we are interested in regularization methods which are optimal on the set $M_{\varphi,E}$ defined by (2.2), that is, regularization methods that guarantee optimal error bounds $\|x_\alpha^\delta - x^\dagger\| \le E\sqrt{\varrho^{-1}(\delta^2/E^2)}$. These regularization methods require the knowledge of the index function φ, the constant E and the noise level δ. In case of source sets (2.2) with power type index functions $\varphi(\lambda) = \lambda^p$, different optimal regularization methods have been given in [32, 33]. For the more general case of index functions $\varphi(\lambda)$ for which $\varrho(t) := t\varphi^{-1}(t)$ is convex, three optimal regularization methods may be found in [30]: the method of Tikhonov regularization of Example 5.1 with $G^2 = \varphi(A^*A)$, the modified spectral method of Example 5.7 with $G = I$ and some further modified spectral method which is similar to that of Example 5.7. In all three methods formulae for the regularization parameter α have been derived leading to optimal error bounds. In our next theorem we consider the modified spectral method of Example 5.7 with $G = I$, apply this method to Problem P1 of Section 1 and show how to choose the regularization parameter such that this method is an optimal regularization method. We provide a new proof which simplifies the general way of proof given in [30] and allows (due to the decomposition idea of Subsection 3.4) more general situations.

Theorem 5.8. *Assume that the solution $u(r,z)$ of the operator equation (1.4) obeys the a priori bound $\|u(\cdot,0)\| \le E$. Let the data at $z = d$ satisfy $\|u(\cdot,d) - u^\delta(\cdot,d)\| \le \delta$ with $\delta \le E$ and let the regularized solution be defined by the modified spectral method (5.2), (5.4). If $\alpha := \alpha_0 < 1$ is chosen by (5.13), then we have the optimal error bound*

$$\|u_\alpha^\delta(\cdot,z) - u(\cdot,z)\| \le \delta \cosh\left(\frac{d-z}{d}\operatorname{arcosh}\frac{E}{\delta}\right). \tag{5.6}$$

Proof. Let $u_\alpha(r,z)$ be the regularized solution (5.2), (5.4) with exact data $\widehat{u}(\xi,d)$ instead of noisy data $\widehat{u}^\delta(\xi,d)$. Then we obtain from (5.4)

$$\widehat{u}_\alpha^\delta(\xi,z) - \widehat{u}_\alpha(\xi,z) = \begin{cases} \left(1/\sqrt{\lambda(|\xi|)}\right)(\widehat{u}^\delta(\xi,d) - \widehat{u}(\xi,d)) & \text{for } \lambda(|\xi|) \ge \alpha \\ (1/\sqrt{\alpha})\,(\widehat{u}^\delta(\xi,d) - \widehat{u}(\xi,d)) & \text{for } \lambda(|\xi|) < \alpha. \end{cases} \tag{5.7}$$

From (5.7) and $\|\widehat{u}^\delta(\cdot,d) - \widehat{u}(\cdot,d)\| \le \delta$ we have the estimate

$$\|\widehat{u}_\alpha^\delta(\cdot,z) - \widehat{u}_\alpha(\cdot,z)\| \le \delta/\sqrt{\alpha}\,. \tag{5.8}$$

Due to (5.4), for the error part $\widehat{u}_\alpha(\xi,z) - \widehat{u}(\xi,z)$ there holds

$$\widehat{u}_\alpha(\xi,z) - \widehat{u}(\xi,z) = \begin{cases} 0 & \text{for } \lambda(|\xi|) \ge \alpha \\ \left(1/\sqrt{\alpha} - 1/\sqrt{\lambda(|\xi|)}\right)\widehat{u}(\xi,d) & \text{for } \lambda(|\xi|) < \alpha. \end{cases}$$

Now we use the decomposing idea of Section 3.4. We have $\widehat{u}_\alpha = \widehat{u}_{1,\alpha} + \widehat{u}_{2,\alpha}$ where $\widehat{u}_{1,\alpha}$, $\widehat{u}_{2,\alpha}$ are the orthogonal projections of $\widehat{u}_\alpha$ onto H_1 and H_2, respectively, and

$$\|\widehat{u}_\alpha(\cdot,z) - \widehat{u}(\cdot,z)\|^2 = \|\widehat{u}_{1,\alpha}(\cdot,z) - \widehat{u}_1(\cdot,z)\|^2 + \|\widehat{u}_{2,\alpha}(\cdot,z) - \widehat{u}_2(\cdot,z)\|^2. \tag{5.9}$$

If $\xi \in W$ then $\lambda(\xi) = 1/\cos^2\Big((d-z)\sqrt{k^2-|\xi|^2}\Big) \geq 1$ and

$$\|\widehat{u}_{2,\alpha}(\cdot,z) - \widehat{u}_2(\cdot,z)\| = 0 \quad \text{for} \quad \alpha < 1. \tag{5.10}$$

On the other hand, for $\xi \in I$, the element $\widehat{u}_1(\xi,d)$ possesses the representation

$$\widehat{u}_1(\xi,d) = \sqrt{\lambda(|\xi|)}\,\widehat{u}_1(\xi,z) = \sqrt{\lambda(|\xi|)}\,\sqrt{\varphi_1(\lambda(|\xi|))}\,\widehat{u}_1(\xi,0)$$

with φ_1 defined by (3.8). Hence, by (5.2) and $\|\widehat{u}_1(\cdot,0)\| \leq E$ we have

$$\|\widehat{u}_{1,\alpha}(\cdot,z) - \widehat{u}_1(\cdot,z)\| \leq E \sup_{0<\lambda<\alpha} \left\{ \Big(1-\sqrt{\lambda}/\sqrt{\alpha}\Big)\sqrt{\varphi_1(\lambda)} \right\}. \tag{5.11}$$

From (5.8), (5.9),(5.10) and (5.11), for $\alpha < 1$ we obtain the estimate

$$\|\widehat{u}_\alpha^\delta(\cdot,z) - \widehat{u}(\cdot,z)\| \leq \sup_{0<\lambda<\alpha} \left\{ \frac{\delta}{\sqrt{\alpha}} + E\left(1-\frac{\sqrt{\lambda}}{\sqrt{\alpha}}\right)\sqrt{\varphi_1(\lambda)} \right\}. \tag{5.12}$$

The function $f(\alpha,\lambda) := \frac{\delta}{\sqrt{\alpha}} + E\left(1-\frac{\sqrt{\lambda}}{\sqrt{\alpha}}\right)\sqrt{\varphi_1(\lambda)}$ possesses one stationary point (α_0,λ_0) which is given by the two equations

$$\lambda_0\varphi_1(\lambda_0) = \frac{\delta^2}{E^2} \quad \text{and} \quad \sqrt{\alpha_0} = \frac{\varphi_1(\lambda_0) + \lambda_0\varphi_1'(\lambda_0)}{\sqrt{\lambda_0}\,\varphi_1'(\lambda_0)}. \tag{5.13}$$

For this regularization parameter $\alpha = \alpha_0$ our estimate (5.12) provides

$$\|\widehat{u}_\alpha^\delta(\cdot,z) - \widehat{u}(\cdot,z)\| \leq \sup_{0<\lambda<\alpha_0} \left\{ \frac{\delta}{\sqrt{\alpha_0}} + E\left(1-\frac{\sqrt{\lambda}}{\sqrt{\alpha_0}}\right)\sqrt{\varphi_1(\lambda)} \right\}. \tag{5.14}$$

For $|\xi| > k$ the function φ_1 is monotonically increasing and the resulting function $\varrho(t) := t\varphi_1^{-1}(t)$ is convex. Exploiting the convexity of ϱ it can be shown that the function $f(\alpha_0,\lambda)$ attains its maximum at $\lambda = \lambda_0$. Hence, from (5.14) there follows

$$\|\widehat{u}_\alpha^\delta(\cdot,z)-\widehat{u}(\cdot,z)\| \leq \frac{\delta}{\sqrt{\alpha_0}}+E\left(1-\frac{\sqrt{\lambda_0}}{\sqrt{\alpha_0}}\right)\sqrt{\varphi_1(\lambda_0)} = E\sqrt{\varphi_1(\lambda_0)} = E\sqrt{\varrho^{-1}(\delta^2/E^2)}.$$

Since $\varrho^{-1}(\lambda) = \lambda\cosh^2\left(\frac{d-z}{d}\,\mathrm{arcosh}\frac{1}{\sqrt{\lambda}}\right)$, this estimate provides (5.6). $\square$

Remark 5.9. The regularization parameter α_0 defined by (5.13) can be given explicitly. By elementary computations we find that

$$\sqrt{\alpha_0} = \frac{\delta\, d\, \sinh\left(\mathrm{arcosh}\frac{E}{\delta}\right)}{\delta\,\cosh\left(\frac{d-z}{d}\,\mathrm{arcosh}\frac{E}{\delta}\right) + E(d-z)\sinh\left(\frac{d-z}{d}\,\mathrm{arcosh}\frac{E}{\delta}\right)}.$$

From this formula we obtain the asymptotic representation

$$\sqrt{\alpha_0} = \frac{2d}{d-z}\left(\frac{\delta}{2E}\right)^{1-z/d}(1+o(1)) \quad \text{for} \ \delta \to 0.$$

5.3. **Order optimal regularization.** A regularized solution x_α^δ is called *order optimal* on the set $M_{\varphi,E}$ if $\|x_\alpha^\delta - x^\dagger\| \le cE\sqrt{\varrho^{-1}(\delta^2/E^2)}$ with some $c \ge 1$. We follow the paper [25] and define a regularized solution for problem (1.4) with noisy data $u^\delta(r,d)$ by the spectral method (5.2), (5.3).

Theorem 5.10. *Let the solution $u(r,z)$ of the operator equation (1.4) obey the a priori bound $\|u(\cdot,0)\| \le E$, let the data at $z=d$ satisfy $\|u(\cdot,d) - u^\delta(\cdot,d)\| \le \delta$ with $\delta \le E$ and let the regularized solution $u_\alpha^\delta(r,z)$ be defined by the spectral method (5.2), (5.3). Let φ_1 be defined by (3.8) and let α be chosen as the unique solution of the equation*

$$\alpha\varphi_1(\alpha) = \delta^2/E^2. \tag{5.15}$$

Then the regularized solution obeys the order optimal error bound

$$\|u_\alpha^\delta(\cdot,z) - u(\cdot,z)\| \le 2\delta \cosh\left(\frac{d-z}{d}\,\mathrm{arcosh}\frac{E}{\delta}\right). \tag{5.16}$$

Proof. Let $u_\alpha(r,z)$ be the regularized solution (5.2), (5.3) with the exact data $\widehat{u}(\xi,d)$ instead of noisy data $\widehat{u}^\delta(\xi,d)$. Then we obtain from (5.3)

$$\widehat{u}_\alpha^\delta(\xi,z) - \widehat{u}_\alpha(\xi,z) = \begin{cases} \left(1/\sqrt{\lambda(|\xi|)}\right)\left(\widehat{u}^\delta(\xi,d) - \widehat{u}(\xi,d)\right) & \text{for } \lambda(|\xi|) \ge \alpha \\ 0 & \text{for } \lambda(|\xi|) < \alpha. \end{cases} \tag{5.17}$$

From (5.17) and $\|\widehat{u}^\delta(\cdot,d) - \widehat{u}(\cdot,d)\| \le \delta$ we have the estimate

$$\|\widehat{u}_\alpha^\delta(\cdot,z) - \widehat{u}_\alpha(\cdot,z)\| \le \delta/\sqrt{\alpha}\,. \tag{5.18}$$

Due to (5.3), for the error part $\widehat{u}(\xi,z) - \widehat{u}_\alpha(\xi,z)$ there holds

$$\widehat{u}(\xi,z) - \widehat{u}_\alpha(\xi,z) = \begin{cases} 0 & \text{for } \lambda(|\xi|) \ge \alpha \\ \left(1/\sqrt{\lambda(|\xi|)}\right)\widehat{u}(\xi,d) & \text{for } \lambda(|\xi|) < \alpha. \end{cases} \tag{5.19}$$

Now we use the decomposing idea of Section 3.4. We have $\widehat{u}_\alpha = \widehat{u}_{1,\alpha} + \widehat{u}_{2,\alpha}$ where $\widehat{u}_{1,\alpha}$, $\widehat{u}_{2,\alpha}$ are the orthogonal projections of $\widehat{u}_\alpha$ onto H_1 and H_2, respectively, and

$$\|\widehat{u}_\alpha(\cdot,z) - \widehat{u}(\cdot,z)\|^2 = \|\widehat{u}_{1,\alpha}(\cdot,z) - \widehat{u}_1(\cdot,z)\|^2 + \|\widehat{u}_{2,\alpha}(\cdot,z) - \widehat{u}_2(\cdot,z)\|^2.$$

If $\xi \in W$, then $\lambda(\xi) = 1/\cos^2\left((d-z)\sqrt{k^2 - |\xi|^2}\right) \ge 1$. Thus,

$$\|\widehat{u}_{2,\alpha}(\cdot,z) - \widehat{u}_2(\cdot,z)\| = 0 \quad \text{for} \quad \alpha < 1. \tag{5.20}$$

On the other hand, for $\xi \in I$, the element $\widehat{u}_1(\xi,d)$ possesses the representation

$$\widehat{u}_1(\xi,d) = \sqrt{\lambda(|\xi|)}\,\widehat{u}_1(\xi,z) = \sqrt{\lambda(|\xi|)}\,\sqrt{\varphi_1(\lambda(|\xi|))}\,\widehat{u}_1(\xi,0)$$

with φ_1 defined by (3.8). Hence, by (5.19), $\|\widehat{u}_1(\cdot,0)\| \le E$ and the monotonicity of φ_1 which is guaranteed for $|\xi| \ge k$ we have

$$\|\widehat{u}_{1,\alpha}(\cdot,z) - \widehat{u}_1(\cdot,z)\| \le E \sup_{0<\lambda<\alpha} \sqrt{\varphi_1(\lambda)} \le E\,\sqrt{\varphi_1(\alpha)}. \tag{5.21}$$

From (5.18), (5.20) and (5.21) we obtain for $\alpha < 1$ the estimate

$$\|\widehat{u}_\alpha^\delta(\cdot,z) - \widehat{u}(\cdot,z)\| \le \frac{\delta}{\sqrt{\alpha}} + E\sqrt{\varphi_1(\alpha)}\,. \tag{5.22}$$

For α chosen as the unique solution of the equation (5.15) we have $\frac{\delta}{\sqrt{\alpha}} = E\sqrt{\varphi_1(\alpha)}$. Moreover, since $\frac{\delta^2}{E^2} = \alpha\varphi_1(\alpha) = \varphi_1(\alpha)\varphi_1^{-1}(\varphi_1(\alpha))$, for ϱ defined by $\varrho(t) = t\varphi_1^{-1}(t)$ we have $\varrho(\varphi_1(\alpha)) = \frac{\delta^2}{E^2}$ and

$$\varphi_1(\alpha) = \varrho^{-1}(\delta^2/E^2). \tag{5.23}$$

Taking into account the explicite form $\varrho^{-1}(\lambda) = \lambda \cosh^2\left(\frac{d-z}{d}\,\mathrm{arcosh}\frac{1}{\sqrt{\lambda}}\right)$ we get (5.16) from (5.22) and (5.23). □

Remark 5.11. The parameter α defined by (5.15) possesses the explicit form

$$\alpha = 1/\cosh^2\left(\frac{d-z}{d}\,\mathrm{arcosh}\frac{E}{\delta}\right). \tag{5.24}$$

For this regularization parameter the regularized solution (5.3) attains the form

$$\widehat{u}^\delta_\beta(\xi, z) = \begin{cases} \cosh\left((d-z)\sqrt{|\xi|^2 - k^2}\right)\, \widehat{u}^\delta(\xi, d) & \text{for} \quad |\xi|^2 \le \beta \\ 0 & \text{for} \quad |\xi|^2 > \beta \end{cases}$$

with β given by $\beta = k^2 + \left(\frac{1}{d}\,\mathrm{arcosh}\frac{E}{\delta}\right)^2$. The regularized solution in [25] has the same form, but with some other β given by $\beta = k^2 + \left(\frac{1}{d}\ln\frac{2E}{\delta}\right)^2$. However note that both β-values are asymptotically equal.

6. **Discrepancy principle**

The use of formula (5.24) for choosing the regularization parameter in the spectral method (5.2), (5.3) requires to know the smoothness of the unknown solution. Generally, both E and the index function φ of the source set $M_{\varphi,E}$ defined by (2.2) are unknown. In this case *a posteriori* rules for choosing the regularization parameter have to be used. One of the most applied *a posteriori* rules for choosing α is the discrepancy principle. Under general source conditions, this principle has well been studied in [16, 20].

6.1. **Discrepancy principle for spectral methods.** Let us consider the spectral methods of Examples 5.6 and 5.7 with $G = I$. In the discrepancy principle the regularization parameter α is chosen such that the discrepancy

$$d(\alpha) := \|Ax^\delta_\alpha - y^\delta\| \tag{6.1}$$

has the order of the noise level δ. The function d defined by (6.1) possesses the following properties:

(1) d obeys the relations $d(0) = 0$ and $\lim\limits_{\alpha\to\infty} d(\alpha) = \|y^\delta\|$.
(2) $d(\alpha)$ is monotonically increasing.

From these properties we conclude that the equation $d(\alpha) = C\delta$ has a unique solution provided d is continuous and $C\delta < \|y^\delta\|$. We assume throughout this section that the noise level δ is sufficiently small such that $C\delta < \|y^\delta\|$ is guaranteed. Then, in the case of continuous functions d defined by (6.1), the discrepancy principle for choosing α can be applied as follows:

Rule R1 (Discrepancy principle for continuous functions $d(\alpha)$)**.** For given constant $C \geq 1$, choose $\alpha = \alpha_D$ as the unique solution of the nonlinear equation

$$d(\alpha) := \|Ax_\alpha^\delta - y^\delta\| = C\delta. \tag{6.2}$$

For the spectral method of Example 5.6 the function $g_\alpha(\lambda)$ is discontinuous with respect to α. As a consequence, $d(\alpha)$ is discontinuous in case of compact operators A with singular system $\{s_i, u_i, v_i\}_{i\in\mathbb{N}}$. In this case we consider the regularized solution

$$x_n^\delta = \sum_{i=1}^{n} \frac{(y^\delta, v_i)}{s_i}\, u_i \tag{6.3}$$

and modify the discrepancy principle has as follows:

Rule R2 (Discrepancy principle for method (6.3))**.** For given constant $C \geq 1$, choose $n = n_D$ as the first integer for which

$$\|Ax_n^\delta - y^\delta\| \leq C\delta < \|Ax_k^\delta - y^\delta\| \quad \text{for } 0 \leq k < n.$$

Theorem 6.1. *Assume that the solution $x^\dagger$ of the equation (1.1) obeys $x^\dagger \in M_{\varphi,E}$ where $M_{\varphi,E}$ is defined by (2.2). Let $\varrho(t) := t\varphi^{-1}(t)$ be convex, let the data $y^\delta \in Y$ satisfy (1.2) and let x_α^δ be defined by the spectral method of Example 5.6. Then, for α chosen by rule R1 or rule R2 with $C > 1$,*

$$\|x_\alpha^\delta - x^\dagger\| \leq (C+1)E\sqrt{\varrho^{-1}\left(\frac{\delta^2}{E^2}\right)} + \frac{E}{C-1}\sqrt{\varrho^{-1}\left(\frac{(C-1)^2\delta^2}{E^2}\right)}. \tag{6.4}$$

Proof. We prove the theorem for rule R1. Let $x_\alpha = g_\alpha(A^*A)A^*y$ be the regularized solution of Example 5.6 with exact data and let $\alpha = \alpha_D$ the regularization parameter chosen by rule R1. Then, from [20, Proposition 4.1] we have

$$\|x_\alpha - x^\dagger\| \leq (C+1)E\sqrt{\varrho^{-1}(\delta^2/E^2)}. \tag{6.5}$$

We introduce the residual function $r_\alpha(\lambda) := 1 - \lambda g_\alpha(\lambda)$ and obtain from (6.2) and the smoothness assumption $x^\dagger \in M_{\varphi,E}$ that

$$C\delta \leq \|r_\alpha(AA^*)(y - y^\delta)\| + \|r_\alpha(AA^*)y\| \leq \delta + E\left\|r_\alpha(A^*A)\sqrt{A^*A\varphi(A^*A)}\right\|.$$

From this estimate we have $(C-1)\delta \leq E\sqrt{\alpha\varphi(\alpha)}$, or equivalently,

$$\frac{\delta}{\sqrt{\alpha}} \leq \frac{E}{C-1}\sqrt{\varrho^{-1}\left(\frac{(C-1)^2\delta^2}{E^2}\right)}. \tag{6.6}$$

From (6.5), the estimate $\|x_\alpha^\delta - x_\alpha\| \leq \delta/\sqrt{\alpha}$ and (6.6) we obtain (6.4). The proof of the theorem for rule R2 in case of compact operators A is similar. □

Remark 6.2. We note that (6.4) can further be estimated by

$$\|x_\alpha^\delta - x^\dagger\| \leq \kappa\, E\sqrt{\varrho^{-1}(\delta^2/E^2)} \quad \text{with} \quad \kappa = C + 1 + \frac{\max\{1, C-1\}}{C-1}$$

from which we see the order optimality of the discrepancy principle. For $C = 2$ we have $\kappa = 4$.

Some sharper estimate can be obtained for the modified spectral method of Example 5.7. This estimate even holds true for $C = 1$. For the modified spectral method of Example 5.7 we have

Theorem 6.3. *Assume that the solution $x^\dagger$ of the equation (1.1) obeys $x^\dagger \in M_{\varphi,E}$ where $M_{\varphi,E}$ is defined by (2.2). Let $\varrho(t) := t\varphi^{-1}(t)$ be convex, let the data $y^\delta \in Y$ satisfy (1.2) and let x_α^δ be defined by the spectral method of Example 5.7. Then, for α chosen by rule R1 with $C \geq 1$,*

$$\|x_\alpha^\delta - x^\dagger\| \leq (C+2)E\sqrt{\varrho^{-1}\left(\delta^2/E^2\right)}.$$

Proof. The proof follows from [20, Theorem 4.3]. □

6.2. **Application to problem P1.** In this subsection we apply the results of the foregoing subsection to the Cauchy problem for the Helmholtz equation and discuss in addition computational aspects for the realization of the discrepancy principle. Applying Theorems 6.1 and 6.3 yields

Corollary 6.4. *Let $k < \frac{\pi}{2d}$ and assume that the solution $u(r,z)$ of the operator equation (1.4) obeys the a priori bound $\|u(\cdot,0)\| \leq E$. Let the data at $z = d$ satisfy $\|u(\cdot,d) - u^\delta(\cdot,d)\| \leq \delta$ with $\delta \leq E$ and let $u_\alpha^\delta(r,z)$ be defined by the spectral method (5.2), (5.3). Then, for α chosen by rule R1 with $C > 1$,*

$$\|u_\alpha^\delta(\cdot,z) - u(\cdot,z)\| \leq \kappa\,\delta\,\cosh\left(\frac{d-z}{d}\,\mathrm{arcosh}\frac{E}{\delta}\right)$$

with $\kappa = C + 1 + \dfrac{\max\{1, C-1\}}{C-1}$.

Corollary 6.5. *Let $k < \frac{\pi}{2d}$ and assume that the solution $u(r,z)$ of the operator equation (1.4) obeys the a priori bound $\|u(\cdot,0)\| \leq E$. Let the data at $z = d$ satisfy $\|u(\cdot,d) - u^\delta(\cdot,d)\| \leq \delta$ with $\delta \leq E$ and let $u_\alpha^\delta(r,z)$ be defined by the modified spectral method (5.2), (5.4). Then, for α chosen by rule R1 with $C = 1$,*

$$\|u_\alpha^\delta(\cdot,z) - u(\cdot,z)\| \leq 3\delta\,\cosh\left(\frac{d-z}{d}\,\mathrm{arcosh}\frac{E}{\delta}\right).$$

Now let us discuss computational aspects of solving the nonlinear equation $\|Ax_\alpha^\delta - y^\delta\| = C\delta$ for the Cauchy problem for the Helmholtz equation. In our studies we will treat the spectral method of Example 5.6. For this regularization method equation (6.2) attains the form

$$f(\alpha) := \int_{\lambda(|\xi|)<\alpha}\left[\widehat{u}^\delta(\xi,d)\right]^2 \mathrm{d}\xi - C^2\delta^2 = 0$$

with $\lambda(|\xi|) = 1/\cosh^2\left((d-z)\sqrt{|\xi|^2 - k^2}\right)$. Clearly, this equation is equivalent to

$$f(\alpha) := \|u^\delta(\cdot,d)\|^2 - \int_{\lambda(|\xi|)\geq\alpha}\left[\widehat{u}^\delta(\xi,d)\right]^2 \mathrm{d}\xi - C^2\delta^2 = 0.$$

We introduce $\beta = k^2 + \left(\frac{1}{d-z}\,\mathrm{arcosh}\frac{1}{\sqrt{\alpha}}\right)^2$ and obtain the equivalent equation

$$f(\beta) := \|u^\delta(\cdot,d)\|^2 - \int_{|\xi|^2\leq\beta}\left[\widehat{u}^\delta(\xi,d)\right]^2 \mathrm{d}\xi - C^2\delta^2 = 0. \tag{6.7}$$

The function f defined by (6.7) possesses the following properties:

(1) f obeys the relations $f(0) = \|u^\delta(\cdot, d)\|^2 - C^2\delta^2$ and $\lim_{\beta\to\infty} f(\beta) = -C^2\delta^2$.

(2) Let $u^\delta(\xi, d)$ be bounded, then f is continuous.

From these properties we conclude that equation (6.7) has a unique solution $\beta = \beta_D$ provided $C\delta < \|u^\delta(\cdot, d)\|$. The regularized solution defined by (5.2), (5.3) with α chosen by rule R1, or equivalently, β chosen by (6.7), can therefore be computed by following steps:

(i) For given data $u^\delta(r, d)$, compute $\|u^\delta(\cdot, d)\|^2$ and $\widehat{u}^\delta(\xi, d) = \mathcal{F}(u^\delta(r, d))$.

(ii) For given $C > 1$, solve the equation (6.7) to obtain $\beta = \beta_D$.

(iii) Compute $\widehat{u}^\delta_\beta(\xi, z)$ according to

$$\widehat{u}^\delta_\beta(\xi, z) = \begin{cases} \cosh\left((d-z)\sqrt{|\xi|^2 - k^2}\right)\, \widehat{u}^\delta(\xi, d) & \text{for} \quad |\xi|^2 \le \beta \\ 0 & \text{for} \quad |\xi|^2 > \beta. \end{cases}$$

(iv) Perform back-transformation to obtain $u^\delta_\beta(r, z) = \mathcal{F}^{-1}\left(\widehat{u}^\delta_\beta(\xi, z)\right)$.

Remark 6.6. We observe one important property of the steps (i) – (iv) for computing the regularized solution $u^\delta_\beta(r, z)$. Namely, the solution $\beta = \beta_D$ of equation (6.7) does not depend on z. As a consequence, if one is interested in the regularized solutions $u^\delta_\beta(r, z)$ for all $z \in [0, d)$, one has to compute the regularization parameter $\beta = \beta_D$ from steps (i) and (ii) only once.

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