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IM PAN Preprint 721 (2010)

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Maximal Commutative Subalgebras, Poisson Geometry and Hochschild Homology

Presented by Piotr M. Hajac

Published as manuscript

Received 05 May 2010

MAXIMAL COMMUTATIVE SUBALGEBRAS, POISSON GEOMETRY AND HOCHSCHILD HOMOLOGY.

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ABSTRACT. A Poisson geometry arising from maximal commutative subalgebras is studied. A spectral sequence convergent to Hochschild homology with coefficients in a bimodule is presented. It depends on the choice of a maximal commutative subalgebra inducing appropriate filtrations. Its $E^2_{p,q}$-groups are computed in terms of canonical homology with values in a Poisson module defined by a given bimodule and a maximal commutative subalgebra.

1. Introduction. In this paper we study the canonical Poisson geometry arising as an effect of choosing a maximal commutative subalgebra in an arbitrary associative algebra. We show that this geometry describes a nonlinear involutive distribution on the spectrum of the maximal commutative subalgebra. Every bimodule over the associative algebra defines a graded sheaf with a flat connection along this nonlinear involutive distribution.

All this can be summarized in the following mental picture based on the idea of noncommutative geometry. Thinking of algebras in the dual manner as of spaces, we consider maximal dominant maps of a given noncommutative space into commutative spaces, "commutative shadows of a noncommutative space". What we see on these commutative shadows is the canonical dynamics. We show that this dynamics on a commutative shadow can be deformed in the sense of Gerstenhaber (quantized) to an almost commutative filtration induced by a given maximal commutative subalgebra of a given noncommutative algebra.

In the paper [1] Jean-Luc Brylinski introduces a spectral sequence convergent to Hochschild homology of an almost commutative algebra (filtered algebra whose associated graded algebra is commutative) and computes (in the case when the associated graded algebra is smooth) its $E^2_{p,q}$-groups in terms of canonical homology of the associated graded algebra. As the main applications serve there envelopping algebras of Lie algebras and algebras of differential operators of commutative algebras. a canonical construction of almost commutative algebras.

After a slight generalization of the result of Brylinski we construct a spectral sequence convergent to Hochschild homology of this almost commutative algebra with coefficients in an almost symmetric bimodule. We compute its $E^2_{p,q}$-groups in terms of canonical Poisson homology of the associated graded Poisson algebra with values in an associated graded Poisson module. We apply this construction to relating Hochschild homology with coefficients on a noncommutative space with canonical Poisson homology on its commutative shadow.

†The author was partially supported by KBN grants:
1P03A 036 26 and 115/E-343/SPB/6.PR UE/DIE 50/2005-2008.

Mathematics Subject Classification (2000): 16E40 , 17B63.

The canonical complex was investigated by Gelfand-Dorfman [3], Koszul [8], Brylinski [3], Huebschmann [6] and Fresse [2] with the relation to Poisson homology. The relations between canonical homology and Poisson geometry were discussed by Vaisman in [12]. The relation between Poisson algebras and Hochschild homology of enveloping algebras was investigated by Kassel [7].

2. The spectral sequence. All rings below are unital and all (bi)modules are unitary. Let R be a noetherian commutative ring of characteristic 0. Unadorned tensor products mean tensor products over R. For an increasing filtration F, $\mathrm{F}_{p-1} \subset \mathrm{F}_p$, we denote its associated gradation Gr, $\mathrm{Gr} = \bigoplus \mathrm{Gr}_p$, $\mathrm{Gr}_p = \mathrm{F}_p/\mathrm{F}_{p-1}$. For any graded abelian group G we denote by G_p its p-th homogeneous part.

Definition 1. A $\mathbb{Z}$-filtered associative R-algebra A (resp. a $\mathbb{Z}$-filtered bimodule over A, symmetric over R), $\mathrm{F}_{p-1}A \subset \mathrm{F}_pA$, $\bigcap_{p\in\mathbb{Z}} \mathrm{F}_pA = 0$, $\bigcup_{p\in\mathbb{Z}} \mathrm{F}_pA = A$ (resp. $\mathrm{F}_{p-1}M \subset \mathrm{F}_pM$, $\bigcap_{p\in\mathbb{Z}} \mathrm{F}_pM = 0$, $\bigcup_{p\in\mathbb{Z}} \mathrm{F}_pM = M$), is called **almost commutative** (resp. **almost symmetric**) if its associated graded algebra $\mathrm{Gr}A$ (resp. associated graded bimodule $\mathrm{Gr}M$) is commutative (resp. symmetric). This means that

$$\mathrm{F}_{p_0}A \cdot \mathrm{F}_{p_1}A \subset \mathrm{F}_{p_0+p_1}A, \quad [\mathrm{F}_{p_0}A, \mathrm{F}_{p_1}A] \subset \mathrm{F}_{p_0+p_1-1}A \tag{1}$$

$$(\text{resp. } \mathrm{F}_{p_0}A \cdot \mathrm{F}_{p_1}M, \mathrm{F}_{p_0}M \cdot \mathrm{F}_{p_1}A \subset \mathrm{F}_{p_0+p_1}A, \quad [\mathrm{F}_{p_0}A, \mathrm{F}_{p_1}M] \subset \mathrm{F}_{p_0+p_1-1}M). \tag{2}$$

On the Hochschild complex $\mathrm{C}_\bullet(A, M) = \bigoplus_k \mathrm{C}_k(A, M)$, $\mathrm{C}_k(A, M) = M \otimes A^{\otimes k}$ we have an increasing filtration F_p

$$\mathrm{F}_p\mathrm{C}_k(A, M) = \sum_{p_0+\cdots+p_k \leq p} \mathrm{F}_{p_0}M \otimes \mathrm{F}_{p_1}A \otimes \cdots \otimes \mathrm{F}_{p_k}A. \tag{3}$$

This gives rise to a spectral sequence with $\mathrm{E}^1_{p,q} = \mathrm{H}_{p+q}(\mathrm{Gr}A, \mathrm{Gr}M)_p$, the homogeneous part of degree p of the Hochschild homology $\mathrm{H}_{p+q}(\mathrm{Gr}A, \mathrm{Gr}M)$, converging to the Hochschild homology $\mathrm{H}_{p+q}(A, M)$. For $M = A$ we obtain the spectral sequence of Brylinski as in [1].

3. Canonical homology.

Definition 2. A commutative graded algebra $B = \bigoplus_{p\in\mathbb{Z}} B_p$, $B_p = 0$ for $p \ll 0$ (resp. a symmetric graded bimodule N over a commutative graded algebra B, $N = \bigoplus_{p\in\mathbb{Z}} N_p$, $N_p = 0$ for $p \ll 0$), is called a **Poisson (graded) algebra** (resp. **Poisson (graded) module** over a (graded) Poisson algebra B) if there is given a Lie algebra structure on B

$$\{-,-\} : B \otimes B \to B$$

(resp. a structure of a right module structure over the Lie algebra $(B, \{-,-\})$

$$\{-,-\} : N \otimes B \to N),$$

(with

$$\{B_{p_0}, B_{p_1}\} \subset B_{p_0+p_1-1} \quad (\text{resp.} \quad \{N_{p_0}, B_{p_1}\} \subset N_{p_0+p_1-1}),$$

if they are Poisson graded) such that for all $b_0, b_1, b_2 \in B$

$$\{b_0, b_1b_2\} = \{b_0, b_1\}b_2 + b_1\{b_0, b_2\}$$

(resp. for all $n \in N, b_1, b_2 \in B$

$$\{nb_1, b_2\} = \{n, b_2\}b_1 + n\{b_1, b_2\}, \quad \{n, b_1b_2\} = \{n, b_1\}b_2 + b_1\{n, b_2\}.)$$

Definition 3. Let N be a Poisson module over a Poisson algebra B. On the graded R-module $\mathrm{C}_\bullet^{can}(B, N) = \bigoplus_k C_k^{can}(B, N)$, $C_k^{can}(B, N) = N \otimes_B \Omega^k_{B/R}$ one defines [2] the chain complex structure as follows:

$$\begin{gathered}\partial : C_k^{can}(B, N) \to C_{k-1}^{can}(B, N),\\ \partial(n \otimes_B \mathrm{d}b_1 \cdots \mathrm{d}b_k) =\\ = \sum_{i=1}^{k} (-1)^{i-1}\{n, b_i\} \otimes_B \mathrm{d}b_1 \cdots \widehat{\mathrm{d}b_i} \cdots \mathrm{d}b_k)\\ + \sum_{1\le i,j\le k} (-1)^{i+j} n \otimes_B \mathrm{d}\{b_i, b_j\}\mathrm{d}b_1 \cdots \widehat{\mathrm{d}b_i} \cdots \widehat{\mathrm{d}b_j} \cdots \mathrm{d}b_k).\end{gathered}$$

One verifies that the boundary operator ∂ is well defined and $\partial^2 = 0$. The homology $\mathrm{H}_\bullet^{can}(B, N)$ of this complex is called **canonical homology** of the Poisson module N over a Poisson algebra B. Note that if B and N are graded Poisson then ∂ is homogeneous of degree (-1). Therefore in the Poisson graded case k-th canonical chain and homology groups are graded in a canonical way.

4. The Hochschild-Kostant-Rosenberg isomorphism. We will use the simple observation [2] that the Hochschild-Kostant-Rosenberg isomorphism (see [1], [5], [9]) holds also in the case of coefficients in a symmetric bimodule N over a smooth commutative algebra B. This means that the map

$$\beta : \mathrm{H}_k(B, N) \to N \otimes_B \Omega^k_{B/R},$$

$$\beta(n \otimes b_1 \otimes \cdots \otimes b_k) = \frac{1}{k!}\, n \otimes_B db_1 \cdots db_k \tag{4}$$

is an isomorphism, with the inverse γ:

$$\gamma(n \otimes_B db_1 \cdots db_k) = \left[\sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}(\sigma)\; n \otimes b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(k)}\right], \tag{5}$$

where the square bracket denotes the Hochschild homology class of a cycle.

5. Hochschild and canonical homology. Given an almost commutative algebra A (resp. an almost symmetric bimodule M over A), $B = \mathrm{Gr}A$ (resp. $N = \mathrm{Gr}M$) has a canonical structure of a graded Poisson algebra (resp. a graded Poisson module over B) with the Poisson structure defined as follows: for $b_0 = a_0 + \mathrm{F}_{p_0-1}A$, $a_0 \in \mathrm{F}_{p_0}A$, $b_1 = a_1 + \mathrm{F}_{p_1-1}A$, $a_1 \in \mathrm{F}_{p_1}A$, $n = m + \mathrm{F}_{p_0-1}M$, $m \in \mathrm{F}_{p_0}M$

$$\begin{gathered}\{b_0, b_1\} = [a_0, a_1] + \mathrm{F}_{p_0+p_1-2}A,\\ \{n, b_1\} = [m, a_1] + \mathrm{F}_{p_0+p_1-2}M.\end{gathered}$$

Moreover we have the following theorem generalizing Theorem 3.1.1 of [1]. In the proof we follow the lines of the beautiful proof of [1], improving a little misprint in the original proof (a_i instead of a_0 in the formula (II) in [1]). At first sight this (spoiled) structure of the formula (II) makes our generalization impossible, but after this minor correction everything can be adapted verbatim.

Theorem 1. *Assume that the above filtrations are bounded and exhaustive, and $B = \mathrm{Gr}A$ is smooth over R. Then for any $q \geq 0$ the Hochschild-Kostant-Rosenberg isomorphism induces an isomorphism of complexes*

$$\beta : (\mathrm{E}^1_{p,q}(A,M), d^1_{p,q}) \to (\mathrm{C}^{can}_{p+q}(B,N)_p, \partial), \tag{6}$$

where $d^1_{p,q} : \mathrm{E}^1_{p,q}(A,M) \to \mathrm{E}^1_{p-1,q}(A,M)$ is the differential in the spectral sequence.
In particular

$$\mathrm{E}^2_{p,q}(A,M) \cong \mathrm{H}^{can}_{p+q}(B,N)_p. \tag{7}$$

Proof. It is enough to prove that $\beta \circ d^1 \circ \gamma = \partial$. Now, the R-module $C^{can}_k(B,N)_p = (N \otimes_B \Omega^k_{B/R})_p$ is generated by elements of the form $n \otimes_B \mathrm{d}b_1 \cdots \mathrm{d}b_k$, where $n = m + \mathrm{F}_{p_0-1}M$, $m \in \mathrm{F}_{p_0}M$, $b_i = a_i + \mathrm{F}_{p_i-1}A$, $a_i \in \mathrm{F}_{p_i}A$, $p_0 + \cdots + p_k = p$. First, we have

$$\gamma(n \otimes_B db_1 \cdots db_k) = \left[\sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma)\ n \otimes b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(k)} \right].$$

The cycle on the right hand side lives in $\mathrm{C}_k(B,N)_p$ and lifts to the chain

$$\sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma)\ m \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)} \in \mathrm{F}_p\mathrm{C}_k(A,M). \tag{8}$$

Its Hochschild boundary is the sum of three terms (I), (II), (III), with

$$\begin{aligned}
(\mathrm{I}) &= \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma)\ m a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \cdots \otimes a_{\sigma(k)}, \\
(\mathrm{II}) &= \sum_{\sigma \in \mathfrak{S}_k} \sum_{1 \leq i < k} \mathrm{sgn}(\sigma)(-1)^i m \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i)} a_{\sigma(i+1)} \otimes \cdots \otimes a_{\sigma(k)}, \\
(\mathrm{III}) &= \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma)(-1)^k a_{\sigma(1)} m \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k-1)}.
\end{aligned}$$

Since the chain (8) is a lift of a Hochschild cycle in $\mathrm{C}_k(B,N)_p$ its Hochschild boundary lives in $\mathrm{F}_{p-1}\mathrm{C}_{k-1}(A,M)$. Now we are to compute the image of this Hochschild boundary in $\mathrm{Gr}_{p-1}\mathrm{C}_{k-1}(A,M) = \mathrm{C}_{k-1}(B,N)_{p-1}$.

First, transforming $\sigma \in \mathfrak{S}_k$ to $\sigma\tau$, where τ is a cyclic permutation, we can rewrite (I) as follows

$$(\mathrm{I}) = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{k+1} \mathrm{sgn}(\sigma)\ m a_{\sigma(k)} \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k-1)}.$$

Since

$$(\mathrm{I}) + (\mathrm{III}) = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{k+1} \mathrm{sgn}(\sigma)\ [m, a_{\sigma(k)}] \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k-1)}$$

we have (I)+(III) $\in \mathrm{F}_{p-1}\mathrm{C}_{k-1}(A,M)$; its image in $\mathrm{Gr}_{p-1}\mathrm{C}_{k-1}(A,M) = \mathrm{C}_{k-1}(B,N)_{p-1}$ is equal to

$$\sum_{\sigma \in \mathfrak{S}_k} (-1)^{k+1} \mathrm{sgn}(\sigma)\ \{n, b_{\sigma(k)}\} \otimes b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(k-1)}. \tag{9}$$

Second, transforming $\sigma \in \mathfrak{S}_k$ to σs_h, where s_h is a transposition which exchanges h and $(h+1)$, we can rewrite (II) as follows

$$(\mathrm{II}) = \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_k} \sum_{1 \leq h < k} \mathrm{sgn}(\sigma)(-1)^h m \otimes a_{\sigma(1)} \otimes \cdots \otimes [a_{\sigma(h)}, a_{\sigma(h+1)}] \otimes \cdots \otimes a_{\sigma(k)}$$

which also lives in $\mathrm{F}_{p-1}\mathrm{C}_{k-1}(A, M)$; its image in $\mathrm{Gr}_{p-1}\mathrm{C}_{k-1}(A, M) = \mathrm{C}_{k-1}(B, N)_{p-1}$ is equal to

$$\frac{1}{2} \sum_{\sigma \in \mathfrak{S}_k} \sum_{1 \leq h < k} \mathrm{sgn}(\sigma)(-1)^h n \otimes b_{\sigma(1)} \otimes \cdots \otimes \{b_{\sigma(h)}, b_{\sigma(h+1)}\} \otimes \cdots \otimes b_{\sigma(k)}. \tag{10}$$

Adding (9) and (10) we obtain $(d^1 \circ \gamma)(n \otimes_B db_1 \cdots db_k)$. It remains to apply β to this.

Now, for the image of the sum (9) under β notice that all σ's with $\sigma(k) = i$ fixed, give the same value for $\beta(\{n, b_{\sigma(k)}\} \otimes b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(k-1)})$ equal to

$$\frac{1}{(k-1)!}(-1)^{k-i}\{n, b_i\} \otimes_B \mathrm{d}b_1 \cdots \widehat{\mathrm{d}b_i} \cdots \mathrm{d}b_{\sigma(k)}.$$

Since there are $(k-1)!$ such permutations, substituting this value, common for each σ with $\sigma(k) = i$, to the image of (9) we obtain

$$\sum_{1 \leq i \leq k} (-1)^{i-1}\{n, b_i\} \otimes_B \mathrm{d}b_1 \cdots \widehat{\mathrm{d}b_i} \cdots \mathrm{d}b_{\sigma(k)}. \tag{11}$$

Next, for the image of (10) under β notice that all pairs (σ, h) with the set $\{\sigma(h), \sigma(h+1)\}$ equal to a fixed set $\{i, j\}$ (say $i < j$), give the same value for $\beta(\mathrm{sgn}(\sigma)(-1)^h n \otimes b_{\sigma(1)} \cdots \otimes \{b_{\sigma(h)}, b_{\sigma(h+1)}\} \otimes \cdots \otimes b_{\sigma(k)})$ equal to

$$\frac{1}{(k-1)!}(-1)^{i+j} n \otimes_B \mathrm{d}\{b_i, b_j\}\mathrm{d}b_1 \cdots \widehat{\mathrm{d}b_i} \cdots \widehat{\mathrm{d}b_j} \cdots \mathrm{d}b_k.$$

Since there are $2(k-1)!$ such pairs, substituting this value, common for each (σ, h) with the set $\sigma(h) = i$ equal to a fixed set, to the image of (10) we obtain

$$\sum_{1 \leq i,j \leq k} (-1)^{i+j} n \otimes_B \mathrm{d}\{b_i, b_j\}\mathrm{d}b_1 \cdots \widehat{\mathrm{d}b_i} \cdots \widehat{\mathrm{d}b_j} \cdots \mathrm{d}b_k. \tag{12}$$

Taking the sum of (11) and (12) we obtain

$$(\beta \circ d^1 \circ \gamma)(n \otimes_B db_1 \cdots db_k) = \partial(n \otimes_B db_1 \cdots db_k). \ \square$$

6. Maximal commutative subalgebras and almost commutative algebras.

Definition 4. Let C be a maximal commutative subalgebra of A and M be a bimodule over A. For $c \in C$, $m \in M$ we define an operation $\mathrm{ad}_c(m) := [c, m]$ and an increasing $\mathbb{N}$-filtration on M

$$\mathrm{F}_p^C M := \{m \in M \mid \forall_{c \in C} \ \ \mathrm{ad}_c^{p+1}(m) = 0\}. \tag{13}$$

Since for all $c, c' \in C$ $[\mathrm{ad}_c, \mathrm{ad}_{c'}] = \mathrm{ad}_{[c,c']} = 0$, the multilinear map

$$(c_0, \ldots, c_p) \mapsto \mathrm{ad}_{c_0} \ldots \mathrm{ad}_{c_p}(m) \tag{14}$$

is symmetric in $(c_0, \ldots, c_p)$. Therefore (14) is a symmetric $(p+1)$-linear form in $(c_0, \ldots, c_p)$ corresponding, via the polarization formula, to the homogeneous polynomial of degree $(p+1)$ in c

$$c \mapsto \mathrm{ad}_c^{p+1}(m). \tag{15}$$

Therefore the filtration F^C can be rewritten equivalently as follows

$$\mathrm{F}_p^C M := \{m \in M \mid \forall_{c_0,\ldots,c_p \in C} \quad \mathrm{ad}_{c_0} \ldots \mathrm{ad}_{c_p}(m) = 0\}. \tag{16}$$

We call $\mathrm{D}^C A := \bigcup_p \mathrm{F}_p^C A$ (resp. $\mathrm{D}^C M := \bigcup_p \mathrm{F}_p^C M$) the **differential hull** of C in A (resp. of the centralizer of C in M). Note that $\mathrm{D}^C M$ inherits the filtration from M and has the same associated gradation.

The following example justifies the name of the differential hull.

Example 1. Let us take two left A-modules P and Q, and form the A-bimodule $M := \mathrm{Hom}_R(P, Q)$. Then by (16) we have

$$\mathrm{F}_p^C M = \mathrm{Diff}_p^{C/R}(P, Q), \quad \mathrm{Gr}_p^C M = \mathrm{Smbl}_p^{C/R}(P, Q),$$

where $\mathrm{Diff}_p^{C/R}(P, Q)$ (resp. $\mathrm{Smbl}_p^{C/R}(P, Q)$) denotes the C-bimodule of differential operators of order p from P to Q (resp. the symmetric C-bimodule of their principal symbols).

Theorem 2. *Given a maximal commutative subalgebra C of A, the above filtration makes $\mathrm{D}^C A$ almost commutative and a bimodule $\mathrm{D}^C M$ over $\mathrm{D}^C A$ almost symmetric.*

Proof. Let $m \in \mathrm{F}_{p_0}^C M$, $a \in \mathrm{F}_{p_1}^C A$.

On the right hand side of the identities

$$\frac{1}{(p_0+p_1+1)!}\mathrm{ad}_c^{p_0+p_1+1}(ma) = \sum_{i+j=p_0+p_1+1} \frac{1}{i!}\mathrm{ad}_c^i(m)\frac{1}{j!}\mathrm{ad}_c^j(a)$$

$$\frac{1}{(p_0+p_1+1)!}\mathrm{ad}_c^{p_0+p_1+1}(am) = \sum_{i+j=p_0+p_1+1} \frac{1}{j!}\mathrm{ad}_c^j(a)\frac{1}{i!}\mathrm{ad}_c^i(m)$$

at least one of the two factors in every summand is zero, since either $i \geq p_0 + 1$ or $j \geq p_1 + 1$. This proves that

$$\mathrm{F}_{p_0}^C M \cdot \mathrm{F}_{p_1}^C A, \ \mathrm{F}_{p_1}^C A \cdot \mathrm{F}_{p_0}^C M \subset \mathrm{F}_{p_0+p_1}^C M. \tag{17}$$

Observe now that $\mathrm{F}_0^C A = C$, since C is a maximal commutative subalgebra in A, and $[\mathrm{F}_0^C M, C] = 0$. Next, if $c \in C$, $m \in \mathrm{F}_p^C M$ then

$$\mathrm{ad}_c^{p+1-i}(\mathrm{ad}_c^i(m)) = \mathrm{ad}_c^{p+1}(m) = 0$$

which means that

$$\mathrm{ad}_c^i(\mathrm{F}_p^C M) \subset \mathrm{F}_{p-i}^C M. \tag{18}$$

On the right hand side of the identity

$$\frac{1}{(p_0+p_1)!}\mathrm{ad}_c^{p_0+p_1+1}([m, a]) = \sum_{i+j=p_0+p_1} \left[\frac{1}{i!}\mathrm{ad}_c^i(m), \frac{1}{j!}\mathrm{ad}_c^j(a)\right]$$

all summands are zero, since the following implications hold:

$$\begin{aligned} i<p_0 \quad &\Rightarrow \quad j>p_1 \;\Rightarrow\; \mathrm{ad}_c^j(a)=0,\\ i=p_0 \quad &\Rightarrow \quad j=p_1 \overset{(18)}{\Rightarrow} \mathrm{ad}_c^i(m)\in \mathrm{F}_0^C M,\ \mathrm{ad}_c^j(a)\in \mathrm{F}_0^C A=C,\\ i>p_0 \quad &\Rightarrow \quad \mathrm{ad}_c^i(m)=0. \end{aligned}$$

This proves that

$$\left[\mathrm{F}^C_{p_0}M,\mathrm{F}^C_{p_1}A,\right]\subset \mathrm{F}^C_{p_0+p_1-1}M. \tag{19}$$

Taking $M=A$ in (17) and (19) we see that the filtration F^C makes $\mathrm{D}^C A$ almost commutative and, also by (17) and (19), $\mathrm{D}^C M$ almost symmetric over $\mathrm{D}^C A$. □

7. Poisson geometry of the differential hull. To discuss the Poisson geometry arising from the differential hull we need to generalize the notion of an involutive distribution and a sheaf with a flat connection along an involutive distribution.

Let C be a smooth commutative R-algebra and let $\Theta^{C/R}=\mathrm{Hom}_C(\Omega^1_{C/R},C)$ denote the relative tangent module of C over R.

Every $f\in \mathrm{Hom}_C(\mathrm{Sym}^p_C\Omega^1_{C/R},C)=\mathrm{Sym}^p_C\Theta^{C/R}$ can be regarded as a symmetric R-linear C-valued p-form on C, which is a derivation with respect to every linear argument. Using this fact we can define the canonical graded Poisson algebra structure of the R-algebra $\mathrm{Sym}_C\Theta^{C/R}=\bigoplus_{p\geq 0}\mathrm{Sym}^p_C\Theta^{C/R}$ of polynomial functions on the relative cotangent bundle of the scheme $\mathrm{Spec}C$ over R as follows: for $f_i\in \mathrm{Hom}_C(\mathrm{Sym}^{p_i}_C\Omega^1_{C/R},C)$, $i=0,1$,

$$(f_0f_1)(c_1,\dots,c_{p_0+p_1}):= \tag{20}$$

$$\frac{1}{(p_0+p_1)!}\sum_{\sigma\in\mathfrak{S}_{p_0+p_1}} f_0(c_{\sigma(1)},\dots,c_{\sigma(p_0)})f_1(c_{\sigma(p_0+1)},\dots,c_{\sigma(p_0+p_1)}),$$

$$\{f_0,f_1\}(c_1,\dots,c_{p_0+p_1-1}):= \tag{21}$$

$$\frac{1}{(p_0+p_1-1)!}\sum_{\sigma\in\mathfrak{S}_{p_0+p_1-1}} (p_0f_0(c_{\sigma(1)},\dots,c_{\sigma(p_0-1)},f_1(c_{\sigma(p_0)},\dots,c_{\sigma(p_0+p_1-1)}))$$

$$-p_1f_1(c_{\sigma(1)},\dots,c_{\sigma(p_1-1)},f_0(c_{\sigma(p_1)},\dots,c_{\sigma(p_0+p_1-1)}))). \tag{22}$$

Definition 5. A ***nonlinear involutive distribution*** on a scheme $\mathrm{Spec}C$ over R is a graded Poisson subalgebra B of the algebra $\mathrm{Sym}_C\Theta^{C/R}$ of polynomial functions on the relative cotangent bundle of the scheme $\mathrm{Spec}C$ over R such that $B_0=C$. A ***graded sheaf with a flat connection*** along the nonlinear involutive distribution B is a graded Poisson module N over B together with a structure of a graded Poisson module over B on $N_0\otimes_C \mathrm{Sym}_C\Theta^{C/R}$ and an embedding of N as a graded Poisson submodule of $N_0\otimes_C\mathrm{Sym}_C\Theta^{C/R}$.

Example 2. Let $L\subset\Theta^{C/R}=\mathrm{Der}_R(C,C)$ be a finitely generated projective C-submodule which is also an R-Lie-subalgebra. It describes an involutive distribution on $\mathrm{Spec}C$ over R. The C-linear embedding $L\hookrightarrow\Theta^{C/R}$ is equivalent to the grading preserving embedding of graded C-algebras

$$\mathrm{Sym}_C L\hookrightarrow \mathrm{Sym}_C\Theta^{C/R}. \tag{23}$$

The Lie subalgebra structure defines on the image of (23) a structure of a graded Poisson subalgebra uniquely determined by the following brackets for $\theta \in \Theta^{C/R}$, $c \in C$, $l \in L$

$$\{\theta, c\} = \theta(c), \quad \{\theta, l\} = [\theta, l]. \tag{24}$$

This shows that an involutive distribution is an instance of a nonlinear involutive distribution. On the other hand, every nonlinear involutive distribution of the form $\mathrm{Sym}_C L$ for some finitely generated projective C-module L defines on L a structure of an involutive distribution. Assume now that N_0 is a C-module equipped with a flat connection along an involutive distribution $L \subset \Theta^{C/R}$

$$\nabla : N_0 \otimes_R L \to N, \quad n_0 \otimes l \mapsto \nabla_l n_0, \tag{25}$$

$$\nabla_{cl} n_0 = c\nabla_l n_0, \quad \nabla_l(n_0 c) = (\nabla_l n_0)c + n_0 l(c), \quad [\nabla_{l'}, \nabla_l] = \nabla_{[l,l']}.$$

Then the brackets

$$\{n_0 \otimes 1, 1 \otimes c\} = 0, \quad \{n_0 \otimes 1, 1 \otimes l\} = -\nabla_l n_0 \otimes 1 \tag{26}$$

determine uniquely structures of graded Poisson modules over $\mathrm{Sym}_C L$ on $N := N_0 \otimes_C \mathrm{Sym}_C L$ and $N_0 \otimes_C \mathrm{Sym}_C \Theta^{C/R}$, and an embedding (of graded Poisson modules over $\mathrm{Sym}_C L$) $N \hookrightarrow N_0 \otimes_C \mathrm{Sym}_C \Theta^{C/R}$. In this way N becomes a graded sheaf with a flat connection along the nonlinear involutive distribution $\mathrm{Sym}_C L$.

Theorem 3. *Let C be a smooth commutative R-subalgebra of an associative R-algebra A and let M be an arbitrary R-symmetric A-bimodule. Then $\mathrm{Gr}^C A$ is a nonlinear involutive distribution on a scheme $\mathrm{Spec} C$ over R and $\mathrm{Gr}^C M$ is a graded sheaf with a flat connection along $\mathrm{Gr}^C A$.*

Proof. Since an A-bimodule M is symmetric as an R-bimodule and C is a commutative R-subalgebra in A M is a left module over $C \otimes_R C$, where $(c_0 \otimes c_1)m := c_0 m c_1$. Let us consider the kernel I of the multiplication map $C \otimes_R C \to C$. This is an ideal in $C \otimes_R C$ generated by elements of the form $c \otimes 1 - 1 \otimes c$. Since $(c \otimes 1 - 1 \otimes c)m = \mathrm{ad}_c(m)$ the filtration (16) gives rise, for any $k \leq p$, to the embedding

$$\mathrm{Gr}^C_p M \hookrightarrow \mathrm{Hom}_C(I^k/I^{k+1}, \mathrm{Gr}^C_{p-k} M), \tag{27}$$

$$m + \mathrm{F}^C_{p-1} M \mapsto$$

$$((c_1 \otimes 1 - 1 \otimes c_1) \cdots (c_k \otimes 1 - 1 \otimes c_k) + I^{k+1} \mapsto \frac{(-1)^k}{k!} \mathrm{ad}_{c_1} \ldots \mathrm{ad}_{c_k}(m) + \mathrm{F}^C_{p-k-1} M).$$

Since for C smooth over R one has the isomorphism of symmetric C-bimodules

$$\mathrm{Sym}^k_C \Omega^1_{C/R} \xrightarrow{\cong} I^k/I^{k+1} \tag{28}$$

and $\mathrm{Gr}^C_0 M$ is a symmetric C-bimodule the latter embedding can be rewritten as

$$\mathrm{Gr}^C_p M \hookrightarrow \mathrm{Hom}_C(\mathrm{Sym}^k_C \Omega^1_{C/R}, \mathrm{Gr}^C_{p-k} M) = \mathrm{Gr}^C_{p-k} M \otimes_C \mathrm{Sym}^k_C \Theta^{C/R}. \tag{29}$$

In particular, since $\mathrm{Gr}^C_0 A = C$, we obtain the embedding

$$\mathrm{Gr}^C_p A \hookrightarrow \mathrm{Hom}_C(\mathrm{Sym}^p_C \Omega^1_{C/R}, C) = \mathrm{Sym}^p_C \Theta^{C/R}. \tag{30}$$

Then the embeddings (30) define a grading preserving embedding of graded Poisson algebras

$$\mathrm{Gr}^C A \hookrightarrow \mathrm{Sym}_C \Theta^{C/R}. \tag{31}$$

The canonical structure of a graded module over the graded algebra $\mathrm{Sym}_C\Theta^{C/R}$ on $\mathrm{Gr}_0^C M \otimes_C \mathrm{Sym}_C\Theta^{C/R}$ is consistent through (31) with the following structure of a graded Poisson module over $\mathrm{Gr}^C A$ on $\mathrm{Gr}_0^C M \otimes_C \mathrm{Sym}_C\Theta^{C/R}$ identified with $\bigoplus_{p\geq 0} \mathrm{Hom}_C(\mathrm{Sym}_C^p \Omega^1_{C/R}, \mathrm{Gr}_0^C M)$: for all $s_0 \in \mathrm{Hom}_C(\mathrm{Sym}_C^{p_0}\Omega^1_{C/R}, \mathrm{Gr}_0^C M))$ and $f_1 \in \mathrm{Hom}_C(\mathrm{Sym}_C^{p_1}\Omega^1_{C/R}, C)$

$$(s_0 f_1)(c_1, \ldots, c_{p_0+p_1}) := \tag{32}$$

$$\frac{1}{(p_0+p_1)!} \sum_{\sigma\in\mathfrak{S}_{p_0+p_1}} s_0(c_{\sigma(1)}, \ldots, c_{\sigma(p_0)}) f_1(c_{\sigma(p_0+1)}, \ldots, c_{\sigma(p_0+p_1)}),$$

$$\{s_0, b_1\}(c_1, \ldots, c_{p_0+p_1-1}) := \tag{33}$$

$$\frac{1}{(p_0+p_1-1)!} \sum_{\sigma\in\mathfrak{S}_{p_0+p_1-1}} (p_0 s_0(c_{\sigma(1)}, \ldots, c_{\sigma(p_0-1)}, b_1(c_{\sigma(p_0)}, \ldots, c_{\sigma(p_0+p_1-1)}))$$
$$+p_1\{s_0(c_{\sigma(p_1)}, \ldots, c_{\sigma(p_0+p_1-1)}), b_1(c_{\sigma(1)}, \ldots, c_{\sigma(p_1-1)}, -)\}),$$

where $\{-,-\}$ on the right hand side is the bracket $\mathrm{Gr}_0^C M \otimes_R \mathrm{Gr}_1^C A \to \mathrm{Gr}_0^C M$. Then the embeddings (29) for $k = p$ define an embedding of graded Poisson modules over $\mathrm{Gr}^C A$

$$\mathrm{Gr}^C M \hookrightarrow \mathrm{Gr}_0^C M \otimes_C \mathrm{Sym}_C\Theta^{C/R}. \tag{34}$$

□

There are also interesting examples of non-smooth maximal commutative subalgebras.

Example 3. Let $A = M_n(k)$ be the $(n \times n)$ matrix algebra over an algebraically closed field k of characteristic 0. Among its many non-isomorphic maximal commutative subalgebras [11] one has the following two extremes:

1) Diagonal subalgebra $C \cong k^n = k \times \cdots \times k$. It is smooth (semisimple) and the filtration stabilizes at C, hence $\mathrm{D}^C A = C$, $\mathrm{Gr}^C A = C$,

2) Subalgebra $C \cong k[x]/(x^{n+1})$ generated by the nilpotent $(n \times n)$ Jordan block. Then the filtration is exhaustive, $\mathrm{D}^C A = \mathrm{Diff}_R C$, the algebra of differential operators of the commutative k-algebra C, and $\mathrm{Gr}^C A$ is isomorphic as a graded Poisson algebra to the polynomial algebra of the nth infinitesimal neighborhood of the nilpotent cone in sl_2^* with its canonical Kirillov-Kostant-Souriau Poisson structure and the grading such that $\deg(e) = 0$, $\deg(h) = 1$, $\deg(f) = 2$ [10].

8. Deformation of the differential hull to a nonlinear involutive distribution. According to the construction of Gerstenhaber [4] we apply a slight generalization of the Rees algebra to our almost commutative algebra $\mathrm{D}^C A$ and an almost symmetric bimodule $\mathrm{D}^C M$.

Definition 6. For any bimodule M over an associative R-algebra A, and a maximal commutative subalgebra C of A, we define

$$\mathrm{Def} M := \sum_p \mathrm{F}_p^C M \cdot t^p \subset M[t] = M \otimes_R R[t]. \tag{35}$$

By Theorem 3, according to [4], we have the following corollary.

Corollary 1. *In the situation of Definition 6 the following holds.*

1) $\mathrm{Def}A$ *is a subalgebra in* $A[t]$ *and* $\mathrm{Def}M$ *is a sub-bimodule of a* $\mathrm{Def}A$*-bimodule* $M[t]$*,*

2) At $t=1$ $\mathrm{Def}A$ *(resp.* $\mathrm{Def}M$*) specializes to the almost commutative algebra* $\mathrm{D}^C A$ *(resp. to the almost symmetric* $\mathrm{Def}A$*-bimodule* $\mathrm{Def}M$*),*

3) At $t=0$ $\mathrm{Def}A$ *(resp.* $\mathrm{Def}M$*) specializes to the nonlinear involutive distribution* $\mathrm{Gr}^C A$ *(resp. to a graded sheaf with a flat connection along* $\mathrm{Gr}^C A$*) on the scheme* $\mathrm{Spec}C$ *over* R*.*

9. Hochschild homology of the differential hull. As a corollary of Theorem 1 and Theorem 2 we obtain the following

Theorem 4. *Let* C *be a maximal commutative subalgebra of an associative* R*-algebra* A *and* M *be a bimodule over* A*. Then*

1) C *defines filtrations* F^C *making* $\mathrm{D}^C A$ *almost commutative and* $\mathrm{D}^C M$ *almost symmetric over* $\mathrm{D}^C A$*, such that* $\mathrm{Gr}^C A$ *is a graded Poisson algebra and* $\mathrm{Gr}^C M$ *a graded Poisson module over* $\mathrm{Gr}^C A$*.*

2) These filtrations give rise to a spectral sequence converging to the Hochschild homology

$$\mathrm{E}^r_{p,q} \Rightarrow \mathrm{H}_{p+q}(\mathrm{D}^C A, \mathrm{D}^C M).$$

3) If $\mathrm{Gr}^C A$ *is smooth over* R *then*

$$\mathrm{E}^2_{p,q} \cong \mathrm{H}^{can}_{p+q}(\mathrm{Gr}^C A, \mathrm{Gr}^C M)_p.$$

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