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Abstract

For solving a linear ill-posed problem, a combination of the Tikhonov regularization and a finite dimensional projection method is considered. In the present paper we treat a parameter describing a discretization level as the second parameter of regularization. The set of all regularization parameter pairs which satisfy a certain discrepancy principle (called the discrepancy set) is investigated. For the case of truncated SVD and LQS projection methods, under the standard source conditions, an order of convergence is derived. It appeared to be the same for all parameters from the discrepancy set. This order of convergence is optimal under the same conditions as those for the Tikhonov regularization with the discrepancy principle and without additional discretization.

Key words. Ill-posed problems, least squares projection method, Tikhonov regularization, two-parameter regularization, discrepancy principle, order optimal error bound.

AMS subject classifications. 65J20, 47A52, 65F22.

1 Introduction

In this paper we consider ill-posed problems of the form

$$Au = f, \tag{1.1}$$

where $A \in L(X, Y)$ is a linear, injective and bounded operator between real infinite dimensional Hilbert spaces X and Y with a nonclosed range $R(A)$. Throughout the paper we assume that $f \in R(A)$ so that (1.1) has a unique solution $u \in X$. Moreover, we assume that f is unknown and a noisy right hand side f^δ with

$$\|f - f^\delta\| \le \delta \tag{1.2}$$

is available only. The stable solving (1.1) with the noisy data f^δ requires an application of appropriate regularization methods. The one of the most widely applied methods is the Tikhonov regularization [1, 2, 11, 12]. Its simplest version consists in solving minimization problem

$$\min_{v \in X} F_\alpha(v), \text{ where } F_\alpha(v) := \|Av - f^\delta\|^2 + \alpha\|v\|^2, \tag{1.3}$$

where $\alpha > 0$ is the regularization parameter which is to be properly chosen. The solution of (1.3) is denoted by u_α^δ.

However, a numerical realization of the method requires a finite dimensional approach. So, in practice the Tikhonov regularization is applied to a finite dimensional discretization of (1.1):

$$A_n u_n^\delta = f_n^\delta,$$

where A_n is a finite dimensional approximation of A. Then the regularization parameter $\alpha = \alpha_n(\delta)$ is chosen for a fixed n (see [1], sec.5.2, [2, 8, 9]). In another approach A_n can be considered as a noisy operator with an error bound $\|A - A_n\| \le h$ and then special regularization methods should be applied to the corresponding linear ill-posed problem with a noisy right hand side and a noisy operator. Such a case has recently been studied in [10, 6, 5].

In this paper we treat the parameter n describing a discretization level as the second parameter of regularization that should be chosen. This approach can be characterized as a special two parameter regularization. The goal of this paper is to describe such a set of parameter pairs for which an optimal order of convergence occurs, when the data noise $\delta \to 0$. The choice of multiple regularization parameters is a crucial and challenging issue for the multi-parameter regularization. For a choice of these two regularization parameters we use the discrepancy principle of the form

$$\|A_n u_{n,\alpha}^\delta - f^\delta\| = C\delta$$

and the set $DS(\delta)$ of all parameters satisfying this principle will be called the discrepancy set.

In the papers [4, 3, 5, 6] multi-parameter regularization was considered in another context: it meant multiple penalty regularization. Some modification of discrepancy principle was used there for the choice of multiple regularization parameters. A corresponding discrepancy curves or surfaces were analyzed and computational aspects were discussed.

The advantage of the multi-parameter regularization is such that it gives more freedom in attaining order-optimal accuracy, since there are many regularization parameters satisfying multi-parameter discrepancy principle. In [4] the authors show one of possible ways to employ this freedom in choosing the regularization. They consider the case when one is interested in an order optimal approximation of the solution u with respect to L_2-norm and simultaneously in an estimation of its value at some point.

In this paper for the case of the combination of the Tikhonov regularization and a finite dimensional projection method we describe the discrepancy set $DS(\delta)$. Under the standard source condition of the form

$$u \in X_{\mu,\rho} := \{v \in X : v = (A^*A)^{\mu} w \text{ and } \|w\| \leq \rho\} \tag{1.4}$$

we prove the optimal order of convergence of $u^{\delta}_{n,\alpha}$ to u, when $\delta \to 0$ and regularization parameter pairs $n(\delta), \alpha(\delta)$ are taken from the discrepancy set $DS(\delta)$, provided $\mu \leq \frac{1}{2}$. The best accuracy that can be guaranteed by the discrepancy principle for such a solution has the order $O(\sqrt{\delta})$. This should be expected, because the simplest version of the Tikhonov regularization (with the qualification equals 1) is used.

In Section 3 the case of discretization by truncated singular value decomposition is taken into account, since it allows us to analyze in details the discrepancy set and its properties. Next, in Section 4 a combination of a least squares projection methods with the Tikhonov regularization is considered. Also in this case, the optimal order of convergence is obtained for regularization parametr pairs belonging to the discrepancy set.

2 Finite dimensional approximation and Tikhonov regularization

For a finite dimensional approximation of (1.1) let us apply projection methods. Let

$$\{X_n\},\ X_n \subset X \text{ and } \{Y_n\},\ Y_n \subset Y$$

be finite dimensional approximations of X and $\overline{R(A)}$, i.e. the union of X_n, is dense in X and the union of Y_n is dense in $\overline{R(A)}$. Let $A_n = Q_n A P_n$, where P_n is the orthogonal projection of X on X_n and Q_n is the orthogonal projection of Y on Y_n. Then (1.1) is approximated by

$$A_n u_n = Q_n f. \tag{2.1}$$

In the case of noisy data, f is replaced by f^{δ} and the solution is denoted by u^{δ}_n.

Natural practical approach to approximate u is to chose the sequence of subspaces $\{X_n\}$ and next to find minimum norm solution of $Au = f$ in a finite dimensional space X_n. In such a case u_n is the minimum norm least squares solution to the equation (2.1) where $A_n = AP_n$ and $Y_n := AX_n$. This method is called LSQ projection method in contrary to the case of dual LSQ method, when the sequence of Y_n is first defined while X_n are related to Y_n: $X_n = A^*Y_n$.

Note that, since A_n has closed range, a least squares solution exists for arbitrary f^δ. Generally, in the case of LSQ projection, without additional assumptions it cannot be guaranteed that $u_n \to u$ as $n \to \infty$ even in the error free case (see in [1] the example given by T.I. Seidman (1980)). Thus a combination of the LSQ projection method with a regularization method should be consider. We focus on the Tikhonov regularization applied to (2.1): the regularized solution $u^\delta_{n,\alpha}$ is the solution of

$$(A_n^* A_n + \alpha)\, u^\delta_{n,\alpha} = A_n^* f^\delta. \tag{2.2}$$

We consider the question of an optimal choice of regularization parameters n and α for given δ. We turn our attention to a-posteriori parameter choice rule. We will consider here the following variant of the widely-used discrepancy principle due to Morozov [7]: We are interested in all parameters (n, α) that fulfill the equation

$$\|Au^\delta_{n,\alpha} - f^\delta\| = C\delta \tag{2.3}$$

for given δ. Therefore we define a discrepancy set $DS(\delta)$ by

$$DS(\delta) := \{n, \alpha :\ n \in N,\ \alpha \in R^+,\ \|Au^\delta_{n,\alpha} - f^\delta\| = C\delta\}. \tag{2.4}$$

This discrepancy set is an analog of discrepancy curve defined and investigated in [3, 4] for multi-parameter regularization of Tikhonov-type, where in the case of two parameters the regularization of solution is defined as the minimizer in X of the functional

$$F_{\alpha,\beta}(v) = \|Av - f^\delta\|^2 + \alpha\|Bv\|^2 + \beta\|x\|^2$$

for a certain operator B.

In the standard approach, the discrepancy principle for Tikhonov regularization of discrete problems has the form

$$\|A_n u^\delta_{n,\alpha} - Q_n f^\delta\| = \widetilde{C}\delta. \tag{2.5}$$

In the case $A_n = AP_n$ considered in the present paper, this formula takes the form: $\|Au^\delta_{n,\alpha} - Q_n f^\delta\| = \widetilde{C}\delta$. Let α^* denote the parameter indicated by (2.4) for a fixed n and a fixed δ. Since $(1 - Q_n)f^\delta \perp AX_n$, we have

$$\|Au^\delta_{n,\alpha} - f^\delta\| = \|Au^\delta_{n,\alpha} - Q_n f^\delta\| + \|(1 - Q_n)f^\delta\|.$$

Thus α^* satisfies the standard discrepancy principle (2.5) with constant

$$\widetilde{C} = C - \frac{\|(1-Q_n)f^\delta\|}{\delta}$$

provided n is sufficiently large such that $\widetilde{C} \geq 1$. Since $\widetilde{\delta} := \widetilde{C}\delta\frac{1}{C} < \delta$, it follows from the above that the parameter α indicated by (2.5) with the same C and n as in (2.3) is greater or equal to α^*.

The reason of introducing the discrepancy principle of the form (2.3) is that we look for the best approximation of the exact solution u, not for the best approximation of the discrete solution u_n.

3 Discrepancy set for the regularized truncated SVD

In order to observe properties of the discrepancy set $DS(\delta)$ in details, we consider first the truncated singular value decomposition as the exceptional projection method which is simultaneously the LSQ and the dual LSQ method.

Let $\{\mu_j, \varphi_j, \psi_j\}_{j=1}^{\infty}$ be the singular system for A where $\mu_1 \geq \mu_2 \geq \cdots$, $A^*A\varphi_j = \mu_j^2\varphi_j$, $A\varphi_j = \mu_j\psi_j$ and $\varphi_j,\ \psi_j$ are normed. Then

$$A = \sum_{j=1}^{\infty} \mu_j(\cdot, \varphi_j)\psi_j \text{ and } u = \sum_{j=1}^{\infty} \frac{f_j}{\mu_j}\varphi, \tag{3.1}$$

where u is the solution of $Au = f$ and $f_j = (f, \psi_j)$.

Let subspaces X_n and Y_n be spanned by $\varphi_j,\ \psi_j$:

$$X_n := span\{\varphi_1, \cdots, \varphi_n\}, \quad Y_n := span\{\psi_1, \cdots, \psi_n\}$$

Then

$$A_n = \sum_{j=1}^{n} \mu_j(\cdot, \varphi_j)\psi_j \tag{3.2}$$

and the solution $u_n^\delta \in X_n$ of the equation $A_n u_n^\delta = Q_n f^\delta$ has the form

$$u_n^\delta = \sum_{j=1}^{n} \frac{f_j^\delta}{\mu_j}\varphi_j \text{ where } f_j^\delta := (f^\delta, \psi_j). \tag{3.3}$$

Since $Y_n = AX_n$ and $X_n = A^*Y_n$, this method is the LSQ as well as the dual LSQ method. It is known that the dual LSQ method has self-regularization property, i.e. $\exists n(\delta)$ such that $\|u_n^\delta - u^\dagger\| \to 0$ as $\delta \to 0$. In our case we have

$$\|u_n^\delta - u^\dagger\| \leq \|P_n u^\dagger - u^\dagger\| + \frac{\delta}{\mu_n},$$

thus in the noise free case u_n is the best possible approximation in X_n. However, in the case of rapidly decreasing singular values (severly ill-problems) finite dimentional problems become very ill conditioned, so we have to combine the projection method with an additional regularization.

On the other hand, the Tikhonov regularization method without discretization yields to regularized solution of the form

$$u_\alpha^\delta = \sum_{j=1}^{\infty} \frac{\mu_j}{\mu_j^2 + \alpha} f_j^\delta \varphi_j. \tag{3.4}$$

It is easy to see that the Tikhonov method applied to the equation $A_n v = Q_n f^\delta$ with the operator (3.2) gives

$$u_{n,\alpha}^\delta = \sum_{j=1}^{n} \frac{\mu_j}{\mu_j^2 + \alpha} f_j^\delta \varphi_j, \tag{3.5}$$

For this case the discrepancy set $DS(\delta)$ defined by (2.4) can be easily described. In order to do that, we need two values: $n(\delta)$ and $\alpha(\delta)$ which are defined by the standard discrepancy principle for the TSVD and the Tikhonov method, respectively. Let $C > 1$ and

$$n(\delta) = \min\{j \in N : \|Au_j^\delta - f^\delta\| \le C\delta\}. \tag{3.6}$$

i.e. for $n = n(\delta)$

$$\|Au_n^\delta - f^\delta\| \le C\delta < \|Au_{n-1}^\delta - f^\delta\|.$$

Let Q be orthogonal projection on $\overline{AX}$. If δ is sufficiently small such that $\|Qf^\delta\| > C\delta$ then $n(\delta)$ exists, since the sequence

$$\phi_j := \|Au_j^\delta - f^\delta\|^2 = \sum_{i=j+1}^{\infty} (f_i^\delta)^2$$

is decreasing to 0 and $\phi_0 \ge \|Qf^\delta\| > C\delta$.

Similarly, let $\alpha(\delta)$ be such that

$$\|Au_{\alpha(\delta)}^\delta - f^\delta\| = C\delta. \tag{3.7}$$

This nonlinear equation has the unique solution if $\|Qf^\delta\| > C\delta$, since from (3.4) the continuous function

$$\phi(\alpha) := \|Au_{\alpha(\delta)}^\delta - f^\delta\|^2 = \sum_{j=1}^{\infty} \left(\frac{\alpha}{\mu_j^2 + \alpha}\right)^2 (f_j^\delta)^2$$

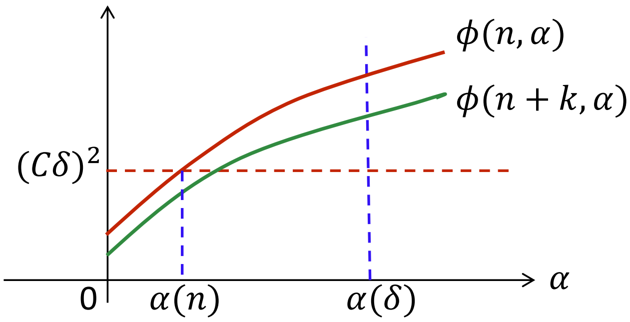

Figure 3.1: Properties of $\phi(n, \alpha)$

is strictly increasing to $\|Qf^\delta\|^2$ when $\alpha \to \infty$ and vanishes at $\alpha = 0$.

Fix $\delta > 0$. Denote

$$\phi(n, \alpha) := \|Au^\delta_{n,\alpha} - f^\delta\|^2.$$

From (3.5) and (3.1) it follows

$$\phi(n, \alpha) = \sum_{j=1}^{n} \frac{\alpha^2}{(\mu_j^2 + \alpha)^2}(f_j^\delta)^2 + \sum_{j=n+1}^{\infty} (f_j^\delta)^2. \tag{3.8}$$

Lemma 3.1 *If $n(\delta)$ and $\alpha(\delta)$ are given by (3.6) and (3.7), respectively, then*

a) $\forall \alpha > 0$ $\phi(n, \alpha) > \phi(n + 1, \alpha)$,

b) $\forall n > 0$ $\phi(n, \alpha)$ *is monotonically increasing w.r to* α,

c) $\forall n \geq n(\delta)$ $\phi(n, 0) \leq (C\delta)^2$,

d) $\forall n$ $\phi(n, \alpha(\delta)) \geq (C\delta)^2$.

Proof: Let us take into account the sequence of functions

$$g_j(\alpha) := \frac{\alpha^2}{(\mu_j^2 + \alpha)^2}.$$

Since $g_j(\alpha) < 1$, from (3.8) the item a) follows. For any $j = 1, 2, \cdots$ and $\alpha > 0$ $g_j'(\alpha) > 0$, so $g_j(\alpha)$ is monotonically increasing and thus $\phi(n, \cdot)$ is

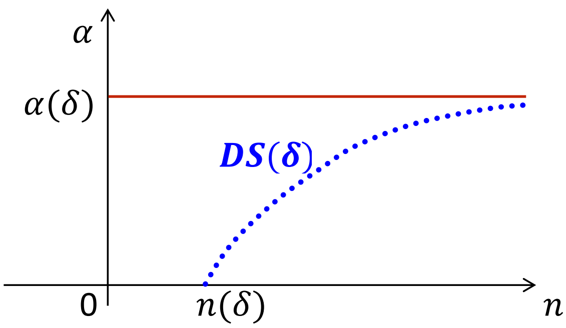

Figure 3.2: Discrepancy set $DS(\delta)$ for the Tikhonov regularization of truncated SVD

monotonically increasing also. Moreover, by the definition of $n(\delta)$, for any $n \geq n(\delta)$

$$\phi(n,0) \leq \sum_{j=n(\delta)+1}^{\infty} (f_j^\delta)^2 = \|Au_n^\delta - f^\delta\|^2 = (C\delta)^2.$$

On the other hand, by (3.8) and the definition of $\alpha(\delta)$ we have

$$\phi(n,\alpha(\delta)) \geq \sum_{j=1}^{\infty} \frac{\alpha^2}{(\mu_j^2+\alpha)^2}(f_j^\delta)^2 = \|Au_\alpha^\delta - f^\delta\|^2 = (C\delta)^2,$$

since $g_j(\alpha) < 1$. ∎

This auxiliary lemma implies the following properties of $DS(\delta)$:

Lemma 3.2 *If $n(\delta)$ and $\alpha(\delta)$ are given by (3.6) and (3.7), respectively, then*

a) *$\forall n > n(\delta)$ there exists the unique $\alpha(n,\delta)$ such that $(n,\alpha(n,\delta)) \in DS(\delta)$*

b) *$\alpha(n,\delta) \to \alpha(\delta)$ as $n \to \infty$.*

Proof: The existence of $\alpha(n,\delta)$ follows directly from the items c) and d) of Lemma 3.1 and its uniqueness follows from the item b). Moreover, from (3.8) $\lim_{n\to\infty}\phi(n,\alpha) = \phi(\alpha)$. Thus, for $\alpha = \alpha(\delta)$ satisfying (3.7)

$$\lim_{n\to\infty}\phi(n,\alpha(\delta)) = \phi(\alpha(\delta)) = (C\delta)^2.$$

On the other hand $\forall n > n(\delta)$ $\phi(n, \alpha(n,\delta)) = (C\delta)^2$. Thus $\alpha(n,\delta) \to \alpha(\delta)$ as $n \to \infty$, which ends the proof. ∎

3.1 Convergence

Let the exact solution of $Au = f$ satisfies the standard source condition (1.4), i.e. $u \in X_{\mu,\rho}$, which means that

$$\sum_{j=1}^{\infty} \frac{f_j^2}{\mu_j^{2+4\mu}} \leq \rho^2. \tag{3.9}$$

We are going to show, that for arbitrary choice of a pair (n, α) from the discrepancy set $DS(\delta)$ the rate of convergence (as $\delta \to 0$) of regularized solution $u_{n,\alpha}^{\delta}$ to u is of the optimal order in the case $\mu \leq \frac{1}{2}$.

We will need the following auxiliary lemma:

Lemma 3.3 *Let $C \geq 1$ in (2.4). If (1.2) and $(n,\alpha) \in DS(\delta)$ then*

$$(C-1)\delta \leq \|Au_{n,\alpha} - f\| \leq (C+1)\delta.$$

PROOF: We have

$$Au_{n,\alpha} - f = \left[Au_{n,\alpha}^{\delta} - f^{\delta}\right] + \left[A(u_{n,\alpha} - u_{n,\alpha}^{\delta}) - (f - f^{\delta})\right]. \tag{3.10}$$

Since $(n,\alpha) \in DS(\delta)$, the norm of the first right hand side term is equal to $C\delta$. Moreover, using the singular value representation of A (3.1) and formula (3.5) we get

$$A(u_{n,\alpha} - u_{n,\alpha}^{\delta}) - (f - f^{\delta}) = \sum_{j=1}^{n} \frac{-\alpha}{\mu^2 + \alpha}(f_j - f_j^{\delta})\psi_j - \sum_{j=n+1}^{\infty}(f_j - f_j^{\delta})\psi_j.$$

Thus, since

$$\frac{\alpha}{\mu_j^2 + \alpha} < 1,$$

the norm of the second term of the right hand side of (3.10) is bounded by δ. Now, applying the standard norm inequalities to (3.10) ends the proof. ∎

The next two Propositions give estimations of the right hand side of the obvious inequality

$$\|u_{n,\alpha}^{\delta} - u\| \leq \|u_{n,\alpha}^{\delta} - u_{n,\alpha}\| + \|u_{n,\alpha} - u\|. \tag{3.11}$$

Proposition 3.4 *Let $u \in X_{\mu,\rho}$ and $\mu \leq \frac{1}{2}$. If (1.2) is satisfied and $(n,\alpha) \in DS(\delta)$ then*

$$\|u^\delta_{n,\alpha} - u_{n,\alpha}\| \leq (C-1)^{-\frac{1}{2\mu+1}} \rho^{\frac{1}{2\mu+1}} \delta^{\frac{2\mu}{2\mu+1}}.$$

Remark If $u \in X_{\mu,\rho}$ and $\mu > \frac{1}{2}$ then for $(n,\alpha) \in DS(\delta)$ we have

$$\|u^\delta_{n,\alpha} - u_{n,\alpha}\| = O(\sqrt{\delta}).$$

Proof: Since

$$u^\delta_{n,\alpha} - u_{n,\alpha} = \sum_{j=1}^{n} \frac{\mu_j}{\mu_j^2 + \alpha} (f_j - f_j^\delta) \varphi_j,$$

from the assumption (1.2) it follows that

$$\|u^\delta_{n,\alpha} - u_{n,\alpha}\| \leq \delta \sup_{1 \leq j \leq n} \frac{\mu_j}{\mu_j^2 + \alpha}. \tag{3.12}$$

The task is now to estimate this supremum. We have

$$Au_{n,\alpha} - f = \sum_{j=1}^{n} \frac{-\alpha}{\mu_j^2 + \alpha} \mu_j^{1+2\mu} \frac{f_j}{\mu_j^{1+2\mu}} \psi_j - \sum_{j=n+1}^{\infty} \mu_j^{1+2\mu} \frac{f_j}{\mu_j^{1+2\mu}} \psi_j.$$

From the source condition (3.9) and from Lemma 3.3 it follows that

$$(C-1)\delta \leq \|Au_{n,\alpha} - f\| \leq \rho \max \left\{ \{ \frac{\alpha}{\mu_j^2 + \alpha} \mu_j^{1+2\mu} \}_{j=1}^n, \mu_{n+1}^{1+2\mu} \right\}.$$

If $\mu + \frac{1}{2} \leq 1$ then

$$\frac{\alpha}{\mu_j^2 + \alpha} (\mu_j^2)^{\mu + \frac{1}{2}} \leq \alpha^{\mu + \frac{1}{2}}.$$

Thus

$$(C-1)\delta \leq \begin{cases} \rho \mu_{n+1}^{2\mu+1}, & \text{as } \alpha \leq \mu_{n+1}^2; \\ \rho \alpha^{\mu + \frac{1}{2}}, & \text{as } \alpha > \mu_{n+1}^2. \end{cases} \tag{3.13}$$

If $\alpha \leq \mu_{n+1}^2$ then $\forall j \leq n$

$$\frac{\mu_j}{\mu_j^2 + \alpha} < \frac{1}{\mu_j} \leq \frac{1}{\mu_{n+1}}.$$

If $\alpha > \mu_{n+1}^2$ then we have two cases:

- if $j \leq n$ such that $\alpha > \mu_j$ then

$$\frac{\mu_j}{\mu_j^2 + \alpha} \leq \frac{\sqrt{\alpha}}{\mu_j^2 + \alpha} \leq \frac{1}{\sqrt{\alpha}};$$

- if $j \leq n$ such that $\alpha > \mu_j$ then

$$\frac{\mu_j}{\mu_j^2 + \alpha} \leq \frac{1}{\mu_j} \leq \frac{1}{\sqrt{\alpha}}.$$

Thus from (3.13) it follows that in all the cases

$$\frac{\mu_j}{\mu_j^2 + \alpha} \leq (C-1)^{-\frac{1}{2\mu+1}} \rho^{\frac{1}{2\mu+1}} \delta^{\frac{-1\mu}{2\mu+1}} \tag{3.14}$$

which due to (3.12) ends the proof. ∎

Proposition 3.5 *If* $u \in X_{\mu,\rho}$ *and* $(n, \alpha) \in DS(\delta)$ *then* $\forall \mu > 0$

$$\|u_{n,\alpha} - u\| \leq (C+1)^{\frac{2\mu}{2\mu+1}} \rho^{\frac{1}{2\mu+1}} \delta^{\frac{2\mu}{2\mu+1}}.$$

Proof: Denote

$$g_j := \begin{cases} \frac{\alpha}{\mu_j^2+\alpha} f_j, & j \leq n; \\ f_j, & j > n. \end{cases}$$

Using (3.1) and (3.5) we get for any $0 \leq \beta < 2$

$$\|u_{n,\alpha} - u\| = \sum_{j=1}^{\infty} \frac{g_j^2}{\mu_j^2} = \sum_{j=1}^{\infty} (\frac{1}{\mu_j^2} g_j^{\beta}) g_j^{2-\beta}.$$

Fix $\beta = \frac{2}{1+2\mu}$. Applying the Hölder inequality with $p = 1 + 2\mu$ and $s = \frac{1+2\mu}{2\mu}$ yields

$$\sum_{j=1}^{\infty} (\frac{1}{\mu_j^2} g_j^{\beta}) g_j^{2-\beta} \leq \left[\sum_{j=1}^{\infty} \frac{1}{\mu_j^{2(1+2\mu)}} g_j^2 \right]^{\frac{1}{1+2\mu}} \left[\sum_{j=1}^{\infty} g_j^2 \right]^{\frac{2\mu}{1+2\mu}}. \tag{3.15}$$

Since $|g_j| \leq |f_j|$,

$$\sum_{j=1}^{\infty} \frac{1}{\mu_j^{2(1+2\mu)}} g_j^2 \leq \sum_{j=1}^{\infty} \frac{1}{\mu_j^{2(1+2\mu)}} f_j^2 \leq \rho^2,$$

which follows from the source condition (3.9). Moreover, since $\|Au_{n,\alpha}-f\|^2 = \sum_{j=1}^{\infty} g_j^2$, from Lemma 3.3 we see that

$$\sum_{j=1}^{\infty} g_j^2 \leq (C+1)^2\delta^2.$$

Substituting these inequalities into (3.15) completes the proof. ∎

Now, we can formulate the main result of this section which follows from Propositions 3.4 and 3.5.

Theorem 3.6 *Suppose that*

- $\|f - f^\delta\| \leq \delta$,
- $u \in X_{\mu,\rho}$,
- $(n, \alpha) \in DS(\delta)$.

If

$$\mu_0 := \min\{\mu, \frac{1}{2}\},$$

then $\exists \widetilde{C} = \widetilde{C}(C, \rho, \mu)$ *such that*

$$\|u_{n,\alpha}^\delta - u\| \leq \widetilde{C}\delta^{\frac{2\mu_0}{2\mu_0+1}}.$$

Remark The combination of the TSVD and the Tikhonov method with regularization parameters from $DS(\delta)$ has the best possible order of accuracy when the source condition (1.4) is satisfied with $\mu \leq \frac{1}{2}$. It follows from the fact that in this combination we apply the classical Tikhonov regularization which has the qualification equals 1.

4 A regularized Ritz approach

This section is devoted to a combination of a least squares projection method with the Tikhonov regularization. Now, contrary to the truncated SVD, the discretization method may be not convergent even for the exact data.

Let $\{X_n\}$ be a finite dimensional approximation of X and let $Y_n := AX_n$. Consider the equation (2.1) with $A_n = AP_n$. The minimizer $u_{n,\alpha}$ of the Tikhonov functional $F_\alpha(z) := \|Az - f\|^2 + \alpha\|z\|^2$ over the subspace X_n satisfies the equation (2.2).

Let us consider first the discrepancy set $DS(\delta)$ (2.4).

Lemma 4.1 *Let C describing the set $DS(\delta)$ be greater than 2. Put*

$$n_0(\delta) := \min\{n : \|f^\delta - Q_n f^\delta\| \leq \delta\}.$$

For any $n \geq n_0(\delta)$ there exists the unique $\alpha = \alpha(n)$ such that $(n, \alpha) \in DS(\delta)$.

Remark We have the following relation between $n_0(\delta)$ and $n(\delta)$ defined by (3.6): $n(\delta) = n_0(C\delta)$. Thus $n(\delta) \leq n_0(\delta)$.

PROOF: From (2.3) the pair $(n, \alpha) \in DS(\delta)$ when

$$C\delta = \|Au^\delta_{n,\alpha} - Q_n f^\delta\| + \|f^\delta - Q_n f^\delta\|.$$

If $n \geq n_0(\delta)$ then $\|f^\delta - Q_n f^\delta\| := c_n(f^\delta)\delta < \delta$ and thus

$$\|Au^\delta_{n,\alpha} - Q_n f^\delta\| = (C - c_n(f^\delta))\delta \text{ where } (C - c_n(f^\delta)) > 1. \tag{4.1}$$

The existence of α follows from the fact that (4.1) is the standard discrepancy principle $\|Au^\delta_{n,\alpha} - Q_n f^\delta\| = \gamma\delta$ with $\gamma > 1$ which has the unique solution $\widetilde{\alpha}(n)$ for any fixed n. ∎

It is easy to check that the auxiliary Lemma 3.3 holds also in this case. Namely, in the proof we replace the singular system $\{\mu_j, \varphi_j, \psi_j\}$ by the singular system of A_n: $\{\mu_{n,j}, \varphi_{n,j}, \psi_{n,j}\}$ and we rewrite the term $f - f^\delta$ in the form

$$\sum_{j=1}^{n} (f_{n,j} - f^\delta_{n,j})\psi_{n,j} + (1 - Q_n)(f - f^\delta).$$

We show here another proof of optimal order of convergence, which is simple, but some additional assumptions are needed, because it is based on the following lemma:

Lemma 4.2 *Let C in (2.4) be greater than 2. Let*

$$m(\delta) := \min\{n : \|A(1 - P_n)u\| \leq \delta\}.$$

Then for any $(n, \alpha) \in DS(\delta)$ such that $n \geq m(\delta)$

$$\|u^\delta_{n,\alpha}\| \leq \|u\|.$$

PROOF: If $(n, \alpha) \in DS(\delta)$ then taking into account that $u^\delta_{n,\alpha}$ is the minimizer of $F_\alpha(z)$ over X_n we get

$$F_\alpha(u^\delta_{n,\alpha}) = (C\delta)^2 + \alpha\|u^\delta_{n,\alpha}\|^2 \leq \|AP_n u - f^\delta\|^2 + \alpha\|P_n u\|^2.$$

Thus

$$\alpha\|u^\delta_{n,\alpha}\|^2 \le \alpha\|u\|^2 + \left[\|AP_n u - f^\delta\|^2 - (C\delta)^2\right].$$

For $n > m(\delta)$ the term in the square brackets is less then zero because

$$\|AP_n u - f^\delta\| \le \|A(1-P_n)u\| + \|f - f^\delta\| \le 2\delta \le C\delta,$$

which ends the proof. ∎

Theorem 4.3 *Let $u \in X_{\mu,\rho}$ and $\mu \le \frac{1}{2}$ and let the assumptions of Lemma 4.2 be satisfied. If $(n,\alpha) \in DS(\delta)$ and $n \ge m(\delta)$ then*

$$\|u^\delta_{n,\alpha} - u\| \le \widetilde{C}\delta^{\frac{2\mu}{2\mu+1}},$$

where $\widetilde{C} = (2\rho(C+1)^{2\mu})^{\frac{1}{2\mu+1}}$.

Proof: By Lemma 4.2 we have

$$\|u^\delta_{n,\alpha} - u\|^2 = \|u^\delta_{n,\alpha}\|^2 - 2(u^\delta_{n,\alpha}, u) + \|u\|^2 \le$$

$$\le 2(\|u\|^2 - (u^\delta_{n,\alpha}, u)) = 2(u - u^\delta_{n,\alpha}, u).$$

Put $u = (A^*A)^\mu w$. Then

$$\|u^\delta_{n,\alpha} - u\|^2 \le 2(u - u^\delta_{n,\alpha}, (A^*A)^\mu w) \le 2\rho\|((A^*A)^{\frac{1}{2}})^{2\mu}(u - u^\delta_{n,\alpha})\|.$$

Using the inequality $\|B^\tau x\| \le \|Bx\|^\tau \|x\|^{1-\tau}$ which holds for a bounded self-adjoint operator B and $0 < \tau \le 1$ and applying it for $B = (A^*A)^{\frac{1}{2}}$ and $\tau = 2\mu_o$ we get

$$\|u^\delta_{n,\alpha} - u\|^2 \le 2\rho\|(A^*A)^{\frac{1}{2}}(u - u^\delta_{n,\alpha})\|^{2\mu}\|u - u^\delta_{n,\alpha}\|^{1-2\mu}. \tag{4.2}$$

Now, let us use the polar decomposition of A: there exists a partially isometric mapping $U : X \to Y$ such that

$$A = U(A^*A)^{\frac{1}{2}} \text{ and } A^* = (A^*A)^{\frac{1}{2}}U.$$

Thus

$$\|(A^*A)^{\frac{1}{2}}A^*\| = \|(A^*A)^{-\frac{1}{2}}(A^*A)^{\frac{1}{2}}U\| = 1$$

and it follows

$$\|(A^*A)^{\frac{1}{2}}(u - u^\delta_{n,\alpha})\| = \|(A^*A)^{-\frac{1}{2}}A^*A(u - u^\delta_{n,\alpha})\| \le \|Au^\delta_{n,\alpha} - f\|.$$

Moreover, if $(n, \alpha) \in DS(\delta)$ then

$$|\|Au_{n,\alpha}^{\delta} - f\| \leq C\delta + \|f - f^{\delta}\| \leq (C+1)\delta.$$

Combining these inequalities with (4.2) we see that

$$\|u_{n,\alpha}^{\delta} - u\|^{1+2\mu} \leq 2\rho(C+1)^{2\mu}\delta^{2\mu}$$

which establishes the assertion of the theorem. ∎

Corollary 4.4 *If $u = (A^*A)^{\mu}w$ and $\|w\| \leq \rho$ with $\mu > \frac{1}{2}$ then $u = (A^*A)^{\frac{1}{2}}v$ where*

$$\|v\| = \|(A^*A)^{\mu-\frac{1}{2}}w\| \leq \rho\|A\|^{2\mu-1} =: \widetilde{\rho}.$$

From this and Theorem 4.3 it follows that for $(n, \alpha) \in DS(\delta)$ and $n \geq m(\delta)$

$$\|u_{n,\alpha}^{\delta} - u\| \leq \sqrt{2\widetilde{\rho}(C+1)}\sqrt{\delta}.$$

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